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On the Duality between Boolean-Valued Analysis and Reduction Theory under the Assumption of Separability

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It is well known that the real and complex numbers in the Scott-Solovay universe $V^{(\mathbf{B})}$ of ZFC based on a complete Boolean algebra **B** are represented by the real-valued and complex-valued Borel functions on the Stonean space Ω of **B**. The main purpose of this paper is to show that the separable complex Hilbert spaces and the von Neumann algebras acting on them in $V^{(\mathbf{B})}$ can be represented by reasonable classes of families of complex Hilbert spaces and of von Neumann algebras over Ω. This could be regarded as the duality between Boolean-valued analysis developed by Ozawa, Takeuti, and others and the traditional reduction theory based not on measure spaces but on Stonean spaces. With due regard to Ozawa, this duality could pass for a sort of reduction theory for AW^*-modules over commutative AW^*-algebras and embeddable AW^*-algebras. Under the duality we establish several fundamental correspondence theorems, including the type correspondence theorems of factors.

1. INTRODUCTION

More than half a century ago Stone (1936) established a duality between Boolean algebras and a certain class of topological spaces. In the 1960s Scott and Solovay used complete Boolean algebras **B** to build $V^{(\mathbf{B})}$ as a model of ZFC and the former established a duality between real numbers in $V^{(\mathbf{B})}$ and real-valued Borel functions on the Stonean space Ω of **B**. This has encouraged some logicians to study more complicated objects in analysis, such as complex Hilbert spaces, von Neumann algebras, etc., by using Boolean-valued models (e.g., Ozawa, 1983, 1984, 1985; Takeuti, 1978, 1983). Such an approach is generally called Boolean-valued analysis.

Operator algebraists have used reduction theory for decomposing such complicated objects as von Neumann algebras into families of simpler

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von Neumann algebras over some appropriately chosen base space. The standard reference for traditional reduction theory is the second part of Dixmier (1981). Some operator algebraists such as Tomita (1953), have recommended Stonean spaces as base spaces of reduction theory.

The main purpose of this paper is to establish a duality between separable complex Hilbert spaces in $V^{(\mathbf{B})}$ and an appropriate class of families of complex Hilbert spaces over Ω and a duality between von Neumann algebras acting on separable complex Hilbert spaces in $V^{(\mathbf{B})}$ and an adequate class of families of von Neumann algebras over Ω. Startlingly enough, the classes of families over Ω in this duality and those studied for a long time in traditional reduction theory coincide! Therefore our duality could be regarded as a duality between Boolean-valued analysis and reduction theory. Since our duality is established under the separability assumption and traditional reduction theory is haunted by separability restrictions, this means that Boolean-valued analysis is indeed a good generalization of traditional reduction theory.

After reviewing Boolean-valued set theory and Scott's duality mentioned above in Sections 2 and 3, respectively, we present our duality for complex Hilbert spaces and von Neumann algebras in Section 4. The duality for von Neumann algebras is much subtler than that for complex Hilbert spaces technically. In Sections 8–10 we are concerned with correspondence results on such properties as types, commutant, intersection, factor, etc., under our duality for von Neumann algebras. Since Tomita–Takesaki theory plays a crucial role in establishing these correspondence results, Sections 5–7 are devoted to Hilbert algebras, unbounded operators, and left Hilbert algebras, respectively. Section 11 shows that, with due regard to Ozawa (1984, 1985), our duality could be regarded as a reduction theory for AW^*-modules over commutative AW^*-algebras and embeddable AW^*-algebras. The last section is devoted to presenting open problems.

Besides a modest acquaintance with Boolean-valued analysis and reduction theory mentioned above, we assume familiarity with the theory of von Neumann algebras up to Tomita–Takesaki theory. For Tomita–Takesaki theory the standard reference is Takesaki (1970). In Section 4 we need a deep result of topological linear spaces, for which the reader is referred to Bourbaki (1953/1955). In Section 9 we need a deep result of Kallman (1971). In Section 1 we need two deep results on the so-called Effros Borel structure due to Effros (1965) and Nielsen (1973), for which a standard reference is Nielsen (1980).

Now we give some miscellaneous remarks. We use $\mathscr{A}$, $\mathscr{M}$, $\mathscr{N}, \ldots,$ with or without indices for von Neumann algebras, Hilbert algebras, and left Hilbert algebras, but the context will prevent any confusion. Similarly, we write $\mathscr{A}'$ for the commutant of $\mathscr{A}$ in the case that $\mathscr{A}$ is a von Neumann algebra and for the associated right Hilbert algebra in the case $\mathscr{A}$ is a left Hilbert algebra. The unit ball of a von Neumann algebra $\mathscr{A}$ is denoted by $(\mathscr{A})_1$. Orthogonal projections on complex Hilbert spaces are usually referred to simply as projections. A linear transformation between complex Hilbert spaces is called an operator if they happen to be the same. A σ-field $\mathbb{S}$ on a set X is called a measurable structure and the pair $(X, \mathbb{S})$ is called a measurable space. A function f from a measurable space $(X, \mathbb{S})$ to a measurable space $(Y, \mathbb{T})$ is called measurable if $f^{-1}(T) \in \mathbb{S}$ for any $T \in \mathbb{T}$.

2. BOOLEAN-VALUED SET THEORY

Let $\mathbf{B}$ be a complete Boolean algebra. We define $V_\alpha^{(\mathbf{B})}$ by transfinite induction on the ordinal α as follows:

1. $V_0^{(\mathbf{B})} = \phi$.
2. $V_\alpha^{(\mathbf{B})} = \{u \mid u: \mathscr{D}(u) \to \mathbf{B} \text{ and } \mathscr{D}(u) \subset \bigcup_{\xi < \alpha} V_\xi^{(\mathbf{B})}\}$.

Then the Boolean-valued universe $V^{(\mathbf{B})}$ of Scott and Solovay is defined as follows:

$$V^{(\mathbf{B})} = \bigcup_{\alpha \in On} V_\alpha^{(\mathbf{B})}$$

where On is the class of all ordinal numbers. The class $V^{(\mathbf{B})}$ can be considered to be a Boolean-valued model of set theory by defining $[\![u \in v]\!]$ and $[\![u = v]\!]$ for $u, v \in V^{(\mathbf{B})}$ with simultaneous induction:

$$[\![u \in v]\!] = \sup_{y \in \mathscr{D}(v)} (v(y) \wedge [\![u = y]\!]) \tag{1}$$

$$[\![u = v]\!] = \inf_{x \in \mathscr{D}(u)} (u(x) \to [\![x \in v]\!]) \wedge \inf_{y \in \mathscr{D}(v)} (v(y) \to [\![y \in u]\!]) \tag{2}$$

and by assigning a Boolean value $[\![\varphi]\!]$ to each formula φ without free variables inductively as follows:

1. $[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$.
2. $[\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$.
3. $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \wedge [\![\varphi_2]\!]$.
4. $[\![\forall x \varphi(x)]\!] = \inf_{u \in V^{(\mathbf{B})}} [\![\varphi(u)]\!]$.
5. $[\![\exists x \varphi(x)]\!] = \sup_{u \in V^{(\mathbf{B})}} [\![\varphi(u)]\!]$.

The following theorem is fundamental to Boolean-valued analysis.

Theorem 2.1. If φ is a theorem of ZFC, then so is $[\![\varphi]\!] = 1$.

The class V of all sets can be embedded into $V^{(\mathbf{B})}$ by transfinite induction as follows:

$$\check{y}=\{(\check{x}, 1)|x\in y\} \qquad \text{for} \quad y\in V$$

Proposition 2.2. For $x, y\in V$, we have

$$[\![\check{x}\in\check{y}]\!]=\begin{cases}1 & \text{if} \quad x\in y\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

$$[\![\check{x}=\check{y}]\!]=\begin{cases}1 & \text{if} \quad x=y\\ 0 & \text{otherwise}\end{cases} \tag{4}$$

A subset $\{b_\alpha\}$ of $\mathbf{B}$ is called a *partition of unity* if $\sup_\alpha b_\alpha=1$ and $b_\alpha\wedge b_\beta=0$ for any $\alpha\neq\beta$. Given a partition of unity $\{b_\alpha\}$ and a subset $\{u_\alpha\}$ of $V^{(\mathbf{B})}$, one can easily prove the following result.

Theorem 2.3. There exists an element u of $V^{(\mathbf{B})}$ such that $[\![u=u_\alpha]\!]\geq b_\alpha$ for any α. Furthermore, this u is determined uniquely in the sense that $[\![u=v]\!]=1$ for any $v\in V^{(\mathbf{B})}$ with the above property.

The above u is denoted by $\sum_\alpha u_\alpha b_\alpha$.

We define the *interpretation* $X^{(\mathbf{B})}$ of $X=\{x\,|\,\varphi(x)\}$ with respect to $V^{(\mathbf{B})}$ to be $\{u\in V^{(\mathbf{B})}|[\![\varphi(u)]\!]=1\}$, assuming that it is not empty. By way of example, $\mathbf{N}^{(\mathbf{B})}$ and $\mathbf{Z}^{(\mathbf{B})}$ stand for the totalities of natural numbers and integers in $V^{(\mathbf{B})}$. For technical convenience, if X is a set, then $X^{(\mathbf{B})}$ is usually considered to be a set by choosing a representative from an equivalence class $\{v\in V^{(\mathbf{B})}|[\![u=v]\!]=1\}$. Then we have $X^{(\mathbf{B})}\times\{1\}\in V^{(\mathbf{B})}$ and

$$[\![X=X^{(\mathbf{B})}\times\{1\}]\!]=1$$

Let $D\subset V^{(\mathbf{B})}$. A function $g\colon D\to V^{(\mathbf{B})}$ is called *extensional* if $[\![d=d']\!]\leq[\![g(d)=g(d')]\!]$ for any $d, d'\in D$. A $\mathbf{B}$-valued set $u\in V^{(\mathbf{B})}$ is said to be *definite* if $u(d)=1$ for any $d\in\mathscr{D}(u)$. Then we have the following characterization theorem of extension maps.

Theorem 2.4. Let $u, v\in V^{(\mathbf{B})}$ be definite and $D=\mathscr{D}(u)$. Then there is a bijective correspondence between $f\in V^{(\mathbf{B})}$ satisfying $[\![f\colon u\to v]\!]=1$ and extensional maps $\varphi\colon D\to v^{(\mathbf{B})}$, where $v^{(\mathbf{B})}=\{u\,|\,[\![u\in v]\!]=1\}$. The correspondence is given by the relation $[\![f(d)=\varphi(d)]\!]=1$ for any $d\in D$.

3. REAL AND COMPLEX NUMBERS

Let $\mathbf{B}$ be a complete Boolean algebra with its Stonean space Ω. Recall that an extremely disconnected compact Hausdorff space is called a Stonean space. We shall fix **B** and Ω throughout this paper. Then, as is well known, there is a bijective correspondence Φ_0 from **B** to the clopen sets of Ω, whose inverse is denoted by Ψ_0. Since the lattice of clopen sets of Ω is isomorphic to the lattice of Borel sets of Ω modulo meager Borel sets, we have the following result.

Theorem 3.1. Each element $b \in \mathbf{B}$ determines a Borel set $\Phi(b)$ up to equivalence, where two Borel sets X, Y are called equivalent if their symmetric difference $X \ominus Y$ is meager. Each Borel set X determines uniquely an element $\Psi(X)$ of **B**. We have $\Psi(\Phi(b)) = b$ for any $b \in \mathbf{B}$, while $\Phi(\Psi(X)) = X$ up to equivalence for any Borel set X of Ω.

This simple duality theorem is the starting point of our duality theory.

The following well-known theorem claims the duality between the complex numbers in $V^{(\mathbf{B})}$ and the complex-valued Borel functions on Ω.

Theorem 3.2. Each complex number r in $V^{(\mathbf{B})}$ determines a complex-valued Borel function $\Phi(r)$ on Ω up to equivalence, where two complex-valued Borel functions on Ω are called equivalent if they are identical except for some meager Borel set of Ω. Each complex-valued Borel function f on Ω determines uniquely a complex number $\Psi(f)$ in $V^{(\mathbf{B})}$. We have $\Psi(\Phi(r)) = r$ for any complex number r in $V^{(\mathbf{B})}$, while $\Phi(\Psi(f)) = f$ up to equivalence for any complex-valued Borel function f on Ω.

A similar duality theorem holds for the real numbers in $V^{(\mathbf{B})}$.

Theorem 3.3. Each real number r in $V^{(\mathbf{B})}$ determines a real-valued Borel function $\Phi(r)$ on Ω up to equivalence, where two real-valued Borel functions on Ω are called equivalent if they are identical except for some meager Borel set of Ω. Each real-valued Borel function f on Ω determines uniquely a real number $\Psi(f)$ in $V^{(\mathbf{B})}$. We have $\Psi(\Phi(r)) = r$ for any real number r in $V^{(\mathbf{B})}$, while $\Phi(\Psi(f)) = f$ up to equivalence for any real-valued Borel function f on Ω.

Although in this paper we do not use measure theory at all, such a convenient expression of measure theory as "for almost all $\omega \in \Omega$" will be used in place of "except for some meager Borel set," since meager Borel sets play a similar role to null sets of measure theory.

Each sequence $\{f_i\}_{i \in \mathrm{N}}$ of complex-valued Borel functions on Ω corresponds to a sequence $\{r_i\}_{i \in \check{\mathrm{N}}}$ of complex numbers in $V^{(\mathbf{B})}$. Then we have the following result.

Theorem 3.4. The sequence $\{r_i\}_{i \in \check{\mathrm{N}}}$ converges to a complex number r in $V^{(\mathbf{B})}$ iff the sequence $\{f_i(\omega)\}_{i \in \mathrm{N}}$ converges to $f(\omega)$ for almost all $\omega \in \Omega$, where $\Phi(r) = f$.

Proof. Essentially the same as that of Takeuti (1978, Part I, Chapter 2, Proposition 2.1, p. 54).

A continuous complex-valued function f defined on a dense open subset $\Omega' \subset \Omega$ is called a *normal function* on Ω if $\lim_{\omega\to\omega'}|f(\omega)| = \infty$ for each $\omega' \in \Omega \backslash \Omega'$. We know that every complex-valued Borel function on Ω is equal almost everywhere to a unique normal function on Ω. For each complex number r in $V^{(\mathbf{B})}$, we denote by $\Phi_0(r)$ the normal function that is equal to $\Phi(r)$ almost everywhere. For more information on normal functions, see Kadison and Ringrose (1983/1986, Section 5.6).

We conclude this section with a technical comment. Usually a subset of a topological space is called Borel if it belongs to the σ-field generated by the open sets, but in the rest of this paper a subset X of the Stonean space Ω shall be called *Borel* (in an extended sense) if there is a Borel set Y (in the usual sense) such that their symmetric difference $X \ominus Y$ is contained in some meager Borel set (in the usual sense), while for other topological spaces, such as the totality $\mathbf{R}$ of real numbers and the totality $\mathbf{C}$ of complex numbers, our notion of a Borel set shall retain the usual sense. Of course, a notion such as a Borel function should be modified accordingly. The reader should notice that the above four theorems are unaffected literally by this modification.

4. COMPLEX HILBERT SPACES AND VON NEUMANN ALGEBRAS

4.1. Hilbert Spaces

A *Borel field of complex Hilbert spaces over* Ω is by definition a family $\{\mathscr{H}(\omega)\}_{\omega\in\Omega}$ of complex Hilbert spaces together with a family $\mathfrak{S}$ of functions on Ω having the following properties:

I. For every $x \in \mathfrak{S}$ and every $\omega \in \Omega$, $x(\omega) \in \mathscr{H}(\omega)$.

II. For every $x \in \mathfrak{S}$, the function $\omega \in \Omega \mapsto \|x(\omega)\|$ is Borel.

III. $\mathfrak{S}$ is a complex linear space with pointwise addition and scalar multiplication.

IV. If y is a function on Ω such that (a) $y(\omega) \in \mathscr{H}(\omega)$ for every $\omega \in \Omega$, and (b) the function $\omega \in \Omega \mapsto \langle x(\omega), y(\omega)\rangle$ is Borel for every $x \in \mathfrak{S}$, then $y \in \mathfrak{S}$.

V. There exists a sequence $\{x_i\}_{i\in\mathbf{N}} \subset \mathfrak{S}$ such that for every $\omega \in \Omega$, the sequence $\{x_i(\omega)\}_{i\in\mathbf{N}}$ is total in $\mathscr{H}(\omega)$ [i.e., the linear span of $\{x_i(\omega)\}_{i\in\mathbf{N}}$ is dense in $\mathscr{H}(\omega)$].

Each function x defined on Ω such that $x(\omega)\in\mathscr{H}(\omega)$ is called a *vector field*.

Each element of $\mathfrak{S}$ is called a *Borel vector field.*

Such a sequence in condition V is called a *fundamental sequence of Borel vector fields.* We notice that for any $x, y\in\mathfrak{S}$, the function $\omega\in\Omega\mapsto\langle x(\omega), y(\omega)\rangle$ is Borel because of the polarization identity

$$4\langle x(\omega), y(\omega)\rangle=\|x(\omega)+y(\omega)\|^2-\|x(\omega)-y(\omega)\|^2$$
$$+i\|x(\omega)+iy(\omega)\|^2-i\|x(\omega)-iy(\omega)\|^2$$

It is also easy to see that if $x\in\mathfrak{S}$, then y is a function on Ω with $y(\omega)\in\mathscr{H}(\omega)$ for every $\omega\in\Omega$, and $x(\omega)=y(\omega)$ almost everywhere, then $y\in\mathfrak{S}$. We notice that the product of a complex-valued Borel function and a Borel vector field is a Borel vector field. It can be seen easily that if a sequence of Borel vector fields converges weakly pointwise on Ω, then its limit is also a Borel vector field.

For each $p=0, 1, \dots, \aleph_0$, we fix once and for all a complex Hilbert space $\mathscr{H}_p$ of dimension p. We regard Ω as the measurable space whose measurable sets are Borel sets, and we regard $\mathscr{H}_p$ as the measurable space induced by the weak or strong topology of $\mathscr{H}_p$. Since $\mathscr{H}_p$ is separable, whether we choose the weaĸ topology or the strong one does not matter. By taking $\mathscr{H}(\omega)$ to be $\mathscr{H}_p$ for any $\omega\in\Omega$ and taking $\mathfrak{S}$ to be the totality of measurable functions from Ω to $\mathscr{H}_p$, we obtain a Borel field of complex Hilbert spaces over Ω, which is called the *constant field* of dimension p and denoted by $\mathfrak{H}_p(\Omega)$.

We notice that our notion of a Borel field of complex Hilbert spaces over Ω is no other than Dixmier's (1981, Part II, Chapter 1) notion of a measurable field of complex Hilbert spaces over a measure space adapted appropriately to our present context. Hence the four propositions of Dixmier (1981, pp. 166–168) carry over to this context with obvious modifications. In particular, given a Borel field $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ of complex Hilbert spaces over Ω, the first three of them go as follows:

Proposition 4.1. (1) For each $p=0, 1, 2, \dots, \aleph_0$, the set $\Omega_p=\{\omega\in\Omega\,|\,$the dimension $d(\omega)$ of $H(\omega)$ is equal to $p\}$ is Borel.

(2) There exists a sequence $\{x_i\}_{i\in\mathbf{N}}$ of Borel vector fields such that the nonzero terms of the sequence $\{x_i(\omega)\}_{i\in\mathbf{N}}$ form an orthonormal basis of $\mathscr{H}(\omega)$ for each $\omega\in\Omega$, i.e., (a) $\langle x_i(\omega), x_j(\omega)\rangle=0$ for any $i\neq j$, (b) $\langle x_i(\omega), x_i(\omega)\rangle=1$ provided $\{x_i(\omega), x_i(\omega)\rangle\neq 0$ for any $i\in\mathbf{N}$ and (c) the sequence $\{x_i(\omega)\}_{i\in\mathbf{N}}$ is total in $\mathscr{H}(\omega)$.

A sequence like $\{x_i(\omega)\}_{i\in\mathbf{N}}$ in part 2 of the above proposition is called an *orthonormal basic sequence* of $\mathscr{H}(\omega)$, while a sequence like $\{x_i\}_{i\in\mathbf{N}}$ in part

2 is called a *Borel field of orthonormal bases*. The latter definition is a bit weaker than that of Dixmier (1981, p. 166).

Proposition 4.2. Let $\{x_i\}_{i\in\mathbf{N}}$ be a fundamental sequence of Borel vector fields. Then a vector field y is a Borel vector field iff the function $\omega\in\Omega\mapsto\langle y(\omega), x_i(\omega)\rangle$ is Borel for any $i\in\mathbf{N}$.

Proposition 4.3. For each $\omega\in\Omega$ there exists a Hilbert space isomorphism $\eta_\omega: \mathscr{H}(\omega)\to\mathscr{H}_{d(\omega)}$ such that for any function y on Ω with $y(\omega)\in\mathscr{H}(\omega)$ for any $\omega\in\Omega$, $y\in\mathfrak{S}$ iff the function $\omega\in\Omega_p\mapsto\eta_\omega(y(\omega))$ is measurable for any $p=0, 1, \ldots, \aleph_0$, where $\mathscr{H}_p$ is endowed with the measurable structure consisting of all Borel subsets of $\mathscr{H}_p$ with respect to the weak or strong topology and Ω_p is endowed with the measurable structure consisting of all Borel subsets of Ω_p.

This proposition claims, roughly speaking, that the constant fields are typical examples of Borel fields of complex Hilbert spaces over Ω.

The last of Dixmier's four propositions mentioned above goes as follows:

Proposition 4.4. Let $\{\mathscr{H}(\omega)\}_{\omega\in\Omega}$ be a family of complex Hilbert spaces indexed by Ω. Let $\{x_i\}_{i\in\mathbf{N}}$ be a sequence of functions on Ω such that (a) $x_i(\omega)\in\mathscr{H}(\omega)$ for any $\omega\in\Omega$ and any $i\in\mathbf{N}$, (b) the function $\omega\in\Omega\mapsto\langle x_i(\omega), x_j(\omega)\rangle$ is Borel for any $i, j\in\mathbf{N}$, and (c) the sequence $\{x_i(\omega)\}_{i\in\mathbf{N}}$ is total in $\mathscr{H}(\omega)$ for any $\omega\in\Omega$. Then there exists a unique Borel field $(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ of complex Hilbert spaces over Ω such that x_i is a Borel vector field for any $i\in\mathbf{N}$.

Two Borel fields $(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ and $(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{T})$ of complex Hilbert spaces over Ω are called *equivalent* if there exist a meager Borel subset Z of Ω and a (Hilbert space) isomorphism $\varphi_\omega: \mathscr{H}(\omega)\to\mathscr{K}(\omega)$ for every $\omega\in\Omega\backslash Z$ such that:

I. For every $x\in\mathfrak{S}$ there exists $y\in\mathfrak{T}$ satisfying $y(\omega)=\varphi_\omega(x(\omega))$ for every $\omega\in\Omega\backslash Z$.

II. For every $y\in\mathfrak{T}$ there exists $x\in\mathfrak{S}$ satisfying $y(\omega)=\varphi_\omega(x(\omega))$ for every $\omega\in\Omega\backslash Z$.

Now we would like to establish the duality between the Borel fields of complex Hilbert spaces over Ω and the separable complex Hilbert spaces in $V^{(\mathbf{B})}$. First of all, we will show that each separable complex Hilbert spaces $\mathscr{H}$ in $V^{(\mathbf{B})}$ naturally yields a Borel field $\Phi(\mathscr{H})$ of complex Hilbert spaces over Ω. Let us define

$$\hat{\mathfrak{S}}=\{x\in V^{(\mathbf{B})}\mid [\![x\in\mathscr{H}]\!]=1\}$$

$$X(\omega)=\{x\in\hat{\mathfrak{S}}\mid \Phi_0(\|x\|)(\omega) \text{ is defined}\}$$

$$Y(\omega)=\{x\in\hat{\mathfrak{S}}\mid \Phi_0(\|x\|)(\omega) \text{ is defined and equal to } 0\}$$

Since $\|\check{r}x+\check{s}y\| \leq |\check{r}|\,\|x\|+|\check{s}|\,\|y\|$ in $V^{(\mathbf{B})}$ for any $r, s \in \mathbf{C}$ and any $x, y \in \hat{\mathfrak{S}}$, it is easy to see that $X(\omega)$ is a complex linear space and $Y(\omega)$ is a linear subspace of $X(\omega)$. Let $\pi_\omega: X(\omega) \to X(\omega)/Y(\omega)$ be the canonical projection. Because of the Cauchy–Schwarz inequality $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ in $V^{(\mathbf{B})}$ for any $x, y \in \hat{\mathfrak{S}}$, we have that for any $\omega \in \Omega$ and any $x, y \in X(\omega)$, $\Phi_0(\langle x, y\rangle)(\omega)$ is defined. Since

$$|\langle x, y\rangle - \langle x', y'\rangle| \leq \|x-x'\|\,\|y\| + \|x'\|\,\|y-y'\|$$

in $V^{(\mathbf{B})}$ for any $x, y, x', y' \in \hat{\mathfrak{S}}$, we have that for any $\omega \in \Omega$ and any $x, y, x', y' \in X(\omega)$, if $\pi_\omega(x) = \pi_\omega(x')$ and $\pi_\omega(y) = \pi_\omega(y')$, then

$$\Phi_0(\langle x, y\rangle)(\omega) = \Phi_0(\langle x', y'\rangle)(\omega)$$

Therefore we can safely define an inner product on $X(\omega)/Y(\omega)$ by

$$\langle \pi_\omega(x), \pi_\omega(y)\rangle = \Phi_0(\langle x, y\rangle)(\omega), \qquad x, y \in X(\omega)$$

The completion of the complex pre-Hilbert space $X(\omega)/Y(\omega)$, which is naturally a complex Hilbert space, is denoted by $\bar{\mathscr{H}}(\omega)$. Let $\bar{\mathfrak{S}}$ be the totality of functions f defined on Ω such that $f(\omega) \in \bar{\mathscr{H}}(\omega)$ for any $\omega \in \Omega$ and there exists $x \in \hat{\mathfrak{S}}$ with $f(\omega) = \pi_\omega(x)$ almost everywhere on Ω. Then it is easy to see that $(\{\bar{\mathscr{H}}(\omega)\}_{\omega \in \Omega}, \bar{\mathfrak{S}})$ satisfies conditions I–III in the definition of a Borel field of complex Hilbert spaces over Ω. Since $\mathscr{H}$ is separable in $V^{(\mathbf{B})}$, there exists an orthonormal basic sequence $\{x_i\}_{i \in \check{\mathbf{N}}}$ of $\mathscr{H}$ in $V^{(\mathbf{B})}$, which, by Theorem 2.4, corresponds externally to the sequence $\{x_i^{\vee}\}_{i \in \mathbf{N}}$. For each $i \in \mathbf{N}$, let f_i be the function defined on Ω such that $f_i(\omega) = \pi_\omega(x_i^{\vee})$ for any $\omega \in \Omega$. For each $\omega \in \Omega$, let $\mathscr{H}(\omega)$ be the closed linear subspace of $\bar{\mathscr{H}}(\omega)$ generated by $\{f_i(\omega)\}_{i \in \mathbf{N}}$. Then it is easy to see that the sequence $\{f_i(\omega)\}_{i \in \mathbf{N}}$ is an orthonormal basic sequence of $\mathscr{H}(\omega)$. Let $\mathfrak{S} = \{f \in \bar{\mathfrak{S}} \mid f(\omega) \in \mathscr{H}(\omega)$ for any $\omega \in \Omega\}$. Then it is easy to see that $(\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$ satisfies conditions I–III and V in the definition of a Borel field of complex Hilbert spaces over Ω. As for the remaining condition IV, we have the following result.

Lemma 4.5. $(\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$ satisfies condition IV in the definition of a Borel field of complex Hilbert spaces over Ω.

Proof. Let g be a function defined on Ω such that $g(\omega) \in \mathscr{H}(\omega)$ for any $\omega \in \Omega$ and the function $\omega \in \Omega \mapsto \langle g(\omega), f_i(\omega)\rangle$, denoted by h_i, is Borel for any $i \in \mathbf{N}$. Since

$$\|g(\omega)\|^2 = \sum_{i \in \mathbf{N}} |\langle g(\omega), f_i(\omega)\rangle|^2$$

for any $\omega \in \Omega$, the function $\omega \in \Omega \mapsto \|g(\omega)\|$, denoted by h, is Borel. By Theorem 2.4, the sequence $\{\Psi(h_i)\}_{i \in \mathbf{N}}$ corresponds internally to a sequence

$\{r_i\}_{i\in\check{N}}$ of complex numbers in $V^{(\mathbf{B})}$. Then, by Theorem 3.4, $\Psi(h)^2=\sum_{i\in\check{N}}|r_i|^2$ in $V^{(\mathbf{B})}$. Therefore the sequence $\sum_{i\in\check{N}} r_i x_i$ converges strongly to an element y of $\mathscr{H}$ in $V^{(\mathbf{B})}$. Since $\langle y, x_i\rangle=r_i$ for any $i\in\check{\mathbf{N}}$ in $V^{(\mathbf{B})}$, $\langle\pi_\omega(y), f_i(\omega)\rangle=\langle g(\omega), f_i(\omega)\rangle$ for any $i\in\mathbf{N}$ almost everywhere on Ω. By Lemma 4.6 to be established below, $\pi_\omega(y)\in\mathscr{H}(\omega)$ almost everywhere on Ω. Since $\{f_i(\omega)\}_{i\in\mathbf{N}}$ is an orthonormal basic sequence of $\mathscr{H}(\omega)$ for any $\omega\in\Omega$, we have $\pi_\omega(y)=g(\omega)$ almost everywhere on Ω, which implies $g\in\mathfrak{S}$. ■

Lemma 4.6. For any $x\in\hat{\mathfrak{S}}$, $\pi_\omega(x)\in\mathscr{H}(\omega)$ almost everywhere on Ω.

Proof. Since $\{x_i\}_{i\in\check{N}}$ is an orthonormal basic sequence of $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have $x=\sum_{i\in\check{N}}\langle x, x_i\rangle x_i$ in $V^{(\mathbf{B})}$, which means, by Theorem 3.4, that $\pi_\omega(x)=\sum_{i\in\mathbf{N}}\langle\pi_\omega(x), f_i(\omega)\rangle f_i(\omega)$ almost everywhere on Ω. ■

Given $x\in\hat{\mathfrak{S}}$, this lemma enables us to choose $f\in\mathfrak{S}$ such that $f(\omega)=\pi_\omega(x)$ almost everywhere on Ω. The function f is denoted by $\Phi(x)$.

Let $\{x_i'\}_{i\in\check{N}}$ be another orthonormal basic sequence of $\mathscr{H}$ in $V^{(\mathbf{B})}$. By replacing $\{x_i\}_{i\in\check{N}}$ by $\{x_i'\}_{i\in\check{N}}$ in the above construction leading to $(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$, we obtain another Borel field $(\{\mathscr{H}'(\omega)\}_{\omega\in\Omega}, \mathfrak{S}')$ of complex Hilbert spaces over Ω. Then Lemma 4.6 gives at once the following result.

Lemma 4.7. $\mathscr{H}(\omega)=\mathscr{H}'(\omega)$ almost everywhere on Ω, so that $(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ and $(\{\mathscr{H}'(\omega)\}_{\omega\in\Omega}, \mathfrak{S}')$ are equivalent.

Proof. By Lemma 4.6, $f_i(\omega)\in\mathscr{H}'(\omega)$ almost everywhere on Ω for any $i\in\mathbf{N}$. Therefore $\mathscr{H}(\omega)\subset\mathscr{H}'(\omega)$ almost everywhere on Ω. Similarly, we have $\mathscr{H}'(\omega)\subset\mathscr{H}(\omega)$ almost everywhere on Ω. Thence $\mathscr{H}(\omega)=\mathscr{H}(\omega')$ almost everywhere on Ω. ■

This justifies our notation $\Phi(\mathscr{H})$ for $(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$, because $\Phi(\mathscr{H})$ is determined uniquely up to equivalence, irrespective of our choice of an orthonormal basic sequence $\{x_i\}_{i\in\check{N}}$ of $\mathscr{H}$.

Conversely, given a Borel field $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ of complex Hilbert spaces over Ω, we would like to obtain a corresponding separable complex Hilbert space in $V^{(\mathbf{B})}$. First of all, we define a definite set $\mathscr{H}_1$ in $V^{(\mathbf{B})}$ as follows:

$$\mathscr{D}(\mathscr{H}_1)=\{\check{x}\mid x\in\mathfrak{S}\}$$

$$\mathscr{H}_1(\check{x})=1$$

We define a binary relation R on $\mathscr{H}_1$ in $V^{(\mathbf{B})}$ as follows:

$$R=\{((x,y)^\vee, e(x,y))\mid x,y\in\mathfrak{S}\}$$

where

$$e(x, y) = \Psi(\{\omega \in \Omega \mid x(\omega) = y(\omega)\})$$

It is easy to see that the binary relation R is an equivalence relation on $\mathscr{H}_1$ in $V^{(\mathbf{B})}$. Therefore we can consider the quotient set of $\mathscr{H}_1$ with respect to the equivalence relation R in $V^{(\mathbf{B})}$, which is denoted by $\mathscr{H}$. For any $x \in \mathfrak{S}$ the equivalence class of $\check{x}$ with respect to R in $V^{(\mathbf{B})}$ is denoted by $\tilde{x}$ or by $\Psi(x)$. We can make the set $\mathscr{H}$ a complex vector space in $V^{(\mathbf{B})}$ as follows:

(a) $\tilde{x} + \tilde{y} = (x+y)^{\sim}$ for any $x, y \in \mathfrak{S}$.

(b) $\Psi(f)\tilde{x} = (fx)^{\sim}$ for any $x \in \mathfrak{S}$ and any complex-valued Borel function f on Ω.

Furthermore, the complex vector space $\mathscr{H}$ in $V^{(\mathbf{B})}$ can be regarded as a complex pre-Hilbert space in $V^{(\mathbf{B})}$ as follows:

$$\langle \tilde{x}, \tilde{y} \rangle = \Psi(g) \qquad \text{for any} \quad x, y \in \mathfrak{S}$$

where g is the complex-valued Borel function $\omega \in \Omega \mapsto \langle x(\omega), y(\omega) \rangle$.

We can say more on $\mathscr{H}$.

Lemma 4.8. $\mathscr{H}$ is a separable complex Hilbert space in $V^{(\mathbf{B})}$.

Proof. Let $\{f_i\}_{i \in \mathbf{N}}$ be a Borel field of orthonormal bases of $\mathfrak{H}$. Let $\{x_i\}_{i \in \check{\mathbf{N}}}$ be a sequence of vectors of $\mathscr{H}$ in $V^{(\mathbf{B})}$ with $x_{\check{i}} = \tilde{f}_i$ ($i \in \mathbf{N}$). Then $\langle f_i(\omega), f_j(\omega) \rangle = 0$ for any $i \neq j$ in $\mathbf{N}$ and any $\omega \in \Omega$, while $\{f_i(\omega), f_i(\omega)\rangle$ is 0 or 1 for any $i \in \mathbf{N}$ and any $\omega \in \Omega$. This implies that in $V^{(\mathbf{B})}$, $\langle x_i, x_j \rangle = 0$ for any $i \neq j$ in $\check{\mathbf{N}}$, while $\langle x_i, x_i \rangle$ is $\check{0}$ or $\check{1}$ for any $i \in \check{\mathbf{N}}$. Since $\{f_i(\omega)\}_{i \in \mathbf{N}}$ is total in $\mathscr{H}(\omega)$ for any $\omega \in \Omega$, $\{f(\omega), f_i(\omega)\rangle = 0$ for any $i \in \mathbf{N}$ implies $f(\omega) = 0$ for any $f \in \mathfrak{S}$ and any $\omega \in \Omega$. Therefore $\{x_i\}_{i \in \check{\mathbf{N}}}$ is total in $\mathscr{H}$ in $V^{(\mathbf{B})}$. Now that the separability of $\mathscr{H}$ in $V^{(\mathbf{B})}$ has just been established, it remains to see the completeness of $\mathscr{H}$ in $V^{(\mathbf{B})}$, for which it suffices to show that in $V^{(\mathbf{B})}$, for any bounded linear functional φ on $\mathscr{H}$, there exists $y \in \mathscr{H}$ such that $\varphi(x) = \langle x, y \rangle$ for any $x \in \mathscr{H}$, since in this case the completeness of the Banach dual space $\mathscr{H}^*$ of $\mathscr{H}$ gives at once the completeness of $\mathscr{H}$ in $V^{(\mathbf{B})}$. Let φ be a bounded linear functional on $\mathscr{H}$ in $V^{(\mathbf{B})}$, so that there exists $r \in \mathbf{R}^{(\mathbf{B})}$ with $|\varphi(x)| \leq r\|x\|$ for any $x \in \mathscr{H}$ in $V^{(\mathbf{B})}$. Let $s_0, \ldots, s_n$ be a finite sequence of rational complex numbers. Recall that a complex number is called rational if it is of the form $u + iv$ with rational numbers u, v. Let $\bar{s}_i$ be the complex-valued Borel function on Ω such that $\bar{s}_i(\omega) = s_i$ if $f_i(\omega)$ is a nonzero vector while $\bar{s}_i(\omega) = 0$ otherwise $(0 \leq i \leq n)$. Let $\{t_i\}_{\check{0} \leq i \leq \check{n}}$ be a finite sequence of complex numbers in $V^{(\mathbf{B})}$ with $t_{\check{i}}=\Psi(\bar{s}_i)$ $(0\leq i\leq n)$. Then it is easy to see that

$$\left\|\sum_{\check{0}\leq i\leq\check{n}} t_i x_i\right\|^2=\Psi\left(\sum_{0\leq i\leq n}|\bar{s}_i|^2\right)$$

and that

$$\Phi_0\left(\Psi\left(\sum_{0\leq i\leq n}|\bar{s}_i|^2\right)\right)(\omega)=\sum_{0\leq i\leq n}|\bar{s}_i(\omega)|^2$$

almost everywhere. Therefore, for almost all $\omega\in\Omega$, if $\Phi_0(r)(\omega)$ is defined, then

$$\Phi_0\left(\varphi\left(\sum_{\check{0}\leq i\leq\check{n}} t_i x_i\right)\right)(\omega)$$

is defined and

$$\left|\Phi_0\left(\varphi\left(\sum_{\check{0}\leq i\leq\check{n}} t_i x_i\right)\right)(\omega)\right|\leq\Phi_0(r)(\omega)\left(\sum_{0\leq i\leq n}|\bar{s}_i(\omega)|^2\right)^{1/2}$$

If the sequence $s_0,\ldots,s_n$ ranges over all finite sequences of rational complex numbers, then $\sum_{0\leq i\leq n}\bar{s}_i(\omega)f_i(\omega)$ ranges over a dense subspace of $\mathscr{H}(\omega)$ for all $\omega\in\Omega$. Thus, for almost all $\omega\in\Omega$, we have a bounded linear functional φ_ω on $\mathscr{H}(\omega)$ such that

$$\varphi_\omega\left(\sum_{0\leq i\leq n}\bar{s}_i(\omega)f_i(\omega)\right)=\Phi_0\left(\varphi\left(\sum_{\check{0}\leq i\leq\check{n}} t_i x_i\right)\right)(\omega)$$

Let g be the function defined on Ω such that if φ_ω is defined, then $g(\omega)\in\mathscr{H}(\omega)$ with $\varphi_\omega(x)=\langle x,g(\omega)\rangle$ for any $x\in\mathscr{H}(\omega)$, while if φ_ω is not defined, then $g(\omega)$ is the zero vector of $\mathscr{H}(\omega)$. Since

$$\langle f_i(\omega),g(\omega)\rangle=\varphi_\omega(f_i(\omega))=\Phi_0(\varphi(x_{\check{i}}))(\omega)$$

almost everywhere on Ω, the function $\omega\in\Omega\mapsto\langle f_i(\omega),g(\omega)\rangle$ is Borel for any $i\in\mathbf{N}$. Thence, by Proposition 4.2, we have $g\in\mathfrak{S}$. It is easy to see that $\langle x_{\check{i}},\tilde{g}\rangle=\varphi(x_{\check{i}})$ in $V^{(\mathbf{B})}$ for any $i\in\mathbf{N}$, which implies that $\tilde{g}$ is no other than the desired element. ■

We denote $\mathscr{H}$ constructed in the above by $\Psi(\mathfrak{H})$.

The following theorem, which follows directly from the definitions, shows that Φ gives a bijective correspondence between separable complex Hilbert spaces in $V^{(\mathbf{B})}$ and Borel fields of complex Hilbert spaces over Ω with its inverse Ψ.

Theorem 4.9. For any separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, the complex Hilbert space $\Psi(\Phi(\mathscr{H}))$ is isomorphic to $\mathscr{H}$ in $V^{(\mathbf{B})}$. For any Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω, the Borel field $\Phi(\Psi(\mathfrak{H}))$ of complex Hilbert spaces over Ω is equivalent to $\mathfrak{H}$.

Proof. To deal with the former statement, let $\{x_i\}_{i\in\check{\mathbf{N}}}$ be an orthonormal basic sequence of $\mathscr{H}$ in $V^{(\mathbf{B})}$ and let $\{f_i\}_{i\in\mathbf{N}}$ be its corresponding Borel field of orthonormal bases of $\Phi(\mathscr{H})$ with $f_i(\omega)=\pi_\omega(x_{\check{i}})$ for any $i\in N$ and any $\omega\in\Omega$.

Then it is easy to see that the sequence $\{y_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ with $y_{\check{i}}=\tilde{f}_i$ for any $i\in\mathbf{N}$ is an orthonormal basic sequence of $\Psi(\Phi(\mathscr{H}))$ in $V^{(\mathbf{B})}$ and that there exists a unique unitary transformation U from $\mathscr{H}$ onto $\Psi(\Phi(\mathscr{H}))$ in $V^{(\mathbf{B})}$ such that $U(x_i)=y_i$ for any $i\in\check{\mathbf{N}}$ in $V^{(\mathbf{B})}$. Now, to deal with the latter statement, let $\{f_i\}_{i\in\mathbf{N}}$ be a Borel field of orthonormal bases of $\mathfrak{H}$ and $\{x_i\}_{i\in\check{\mathbf{N}}}$ its corresponding orthonormal basic sequence of $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$ with $x_{\check{i}}=\tilde{f}_i$ for any $i\in\mathbf{N}$. Let $\{g_i\}_{i\in\mathbf{N}}$ be the sequence of Borel vector fields of $\Phi(\Psi(\mathfrak{H}))$ with $g_i(\omega)=\pi_\omega(x_{\check{i}})$ for any $\omega\in\Omega$ and any $i\in\mathbf{N}$. Then it is easy to see that $\langle f_i(\omega), f_j(\omega)\rangle=\langle g_i(\omega), g_j(\omega)\rangle$ almost everywhere for any $i, j\in\mathbf{N}$. Therefore, for almost all $\omega\in\Omega$, there exists a unitary transformation φ_ω of $\mathscr{H}(\omega)$ onto $\mathscr{K}(\omega)$, the totality of which establishes the equivalence of $\mathfrak{H}$ and $\Phi(\Psi(\mathfrak{H}))$, where

$$\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$$

and

$$\Phi(\Psi(\mathfrak{H}))=(\{\mathscr{K}(\omega)\}_{\omega\in\Omega}, \mathfrak{T}) \quad \blacksquare$$

We close this subsection with the following theorem, which has a close relationship with Proposition 4.3.

Theorem 4.10. Let $\mathscr{H}$ be a separable complex Hilbert space in $V^{(\mathbf{B})}$ with

$$\dim(\mathscr{H})=\sum_{0\le n\le\aleph_0}\check{n}b_n$$

Let $\Phi(\mathscr{H})=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$, $B_n=\Phi_0(b_n)$, and $\mathfrak{S}_n=\{$the restriction of x to $B_n\,|\,x\in\mathfrak{S}\}$. Then $(\{\mathscr{H}(\omega)\}_{\omega\in B_n}, \mathfrak{S}_n)$ is a Borel field of complex Hilbert spaces over B_n, which is equivalent to the constant field of dimension n.

Proof. Let $\{x_i\}_{i\in\check{\mathbf{N}}}$ be an orthonormal basic sequence of $\mathscr{H}$ in $V^{(\mathbf{B})}$ and $\{f_i\}_{i\in\mathbf{N}}$ its corresponding sequence of Borel vector fields of $\Phi(\mathscr{H})$ with $f_i(\omega)=\pi_\omega(x_{\check{i}})$ for any $i\in\mathbf{N}$ and any $\omega\in\Omega$. Then it is easy to see that the number of nonzero terms in $\{f_i(\omega)\}_{i\in\mathbf{N}}$ is equal to n for almost all $\omega\in B_n$ and that the nonzero terms in $\{f_i(\omega)\}_{i\in\mathbf{N}}$ gives an orthonormal basis of $\mathscr{H}(\omega)$ for almost all $\omega\in B_n$, from which the desired conclusion follows readily. $\blacksquare$

4.2. Bounded Operators

Let $\mathfrak{H} = (\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω. A family $\{T(\omega)\}_{\omega \in \Omega}$ of bounded operators $T(\omega)$ on $\mathscr{H}(\omega)$ is called a *field of bounded operators* on $\mathfrak{H}$. It is called *Borel* if it satisfies the following condition:

(#) For any Borel vector field x, the vector field

$$\omega \in \Omega \mapsto T(\omega)x(\omega)$$

is Borel.

For any field $\mathfrak{T} = \{T(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}$ and $x \in \mathfrak{S}$, the vector field $\{T(\omega)x(\omega)\}_{\omega \in \Omega}$ is denoted by $\mathfrak{T}x$.

Proposition 4.11. For any Borel field $\{T(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}$, the function $\omega \in \Omega \mapsto \|T(\omega)\|$ is Borel.

Proof. Let $\mathbb{S}$ be the totality of sequences $\{s_i\}_{i \in \mathbf{N}}$ of rational complex numbers in which the s_i are all zero except for a finite number of them and $\sum_{i \in \mathbf{N}} |s_i|^2 \leq 1$.

It is easy to see that $\mathbb{S}$ is a countable set. Let $\{x_i\}_{i \in \mathbf{N}}$ be a Borel field of orthonormal bases. Then we can see readily that

$$\|T(\omega)\| = \sup\left\{ \left\| T(\omega)\left[\sum_{i \in \mathbf{N}} s_i x_i(\omega) \right] \right\| \,\middle|\, \{s_i\}_{i \in \mathbf{N}} \in \mathbb{S} \right\}$$

for any $\omega \in \Omega$. Since the function $\omega \in \Omega \mapsto \|T(\omega)(\sum_{i \in \mathbf{N}} s_i x_i)\|$ is Borel for any $\{s_i\}_{i \in \mathbf{N}} \in \mathbb{S}$, we are now sure that the function $\omega \in \Omega \mapsto \|T(\omega)\|$ is also Borel. ■

We notice that our notion of a Borel field of bounded operators is no other than Dixmier's (1981, Part II, Chapter 2) notion of a measurable field of continuous linear mappings adapted appropriately to our present context. Thus all elementary properties of his notion carry over to our present context with obvious modifications. In particular, we now record a proposition of Dixmier (1981, Part II, Chapter 2, Proposition 1) with some minor modifications.

Proposition 4.12. Let $\{x_i\}_{i \in \mathbf{N}}$ be a fundamental sequence of Borel vector fields of $\mathfrak{H}$. For a field $\{T(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}$ to be Borel, it is necessary and sufficient that the functions $\omega \in \Omega \mapsto \langle T(\omega)x_i(\omega), x_j(\omega)\rangle$ are Borel $(i, j \in \mathbf{N})$.

Two Borel fields $\{T(\omega)\}_{\omega \in \Omega}$, $\{W(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}$ are called *equivalent* if $T(\omega) = W(\omega)$ for almost all $\omega \in \Omega$.

In the preceding subsection we have established a correspondence between the separable complex Hilbert spaces in $V^{(\mathbf{B})}$ and the Borel fields of complex Hilbert spaces over Ω. The principal objective of this subsection is to establish, under this correspondence, an essentially bijective correspondence between the bounded operators in $V^{(\mathbf{B})}$ and the Borel fields of bounded operators. First of all, given a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ and a bounded operator T on $\mathscr{H}$, we are going to construct a Borel field $\Phi(T)$ of bounded operators on the Borel field $\Phi(\mathscr{H})$ of complex Hilbert spaces over Ω. We use the same notation as that used in the preceding subsection for the construction of $\Phi(\mathscr{H})$ from $\mathscr{H}$. It is easy to see that for any $\omega\in\Omega$, if $\Phi_0(\|T\|)(\omega)$ is defined and $x\in X(\omega)$, then $Tx\in X(\omega)$ and $\Phi_0(\|Tx\|)(\omega)\leq\Phi_0(\|T\|)(\omega)\Phi_0(\|x\|)(\omega)$, since $\|Tx\|\leq\|T\|\,\|x\|$ in $V^{(\mathbf{B})}$. Therefore, for any $\omega\in\Omega$ such that $\Phi_0(\|T\|)(\omega)$ is defined, the function $x\in X(\omega)\mapsto Tx$ naturally induces a bounded operator $\bar{T}(\omega)$ on $\bar{\mathscr{H}}(\omega)$. Let $\mathbb{R}$ be the totality of all sequences $\{r_i\}_{i\in\mathbf{N}}$ of rational complex numbers r_i in which the r_i are all zero except for a finite number of $i\in\mathbf{N}$. It is easy to see that $\mathbb{R}$ is a countable set and the set $\{\sum_{i\in\mathbf{N}} r_i f_i(\omega)\,|\,\{r_i\}_{i\in\mathbf{N}}\in\mathbb{R}\}$ is a dense linear subspace of $\mathscr{H}(\omega)$ for any $\omega\in\Omega$. For $\{r_i\}_{i\in\mathbf{N}}\in\mathbb{R}$, we let $\{s_i\}_{i\in\check{\mathbf{N}}}$ be a sequence of complex numbers in $V^{(\mathbf{B})}$ such that $s_{\check{i}}=\check{r}_i$ for any $i\in\mathbf{N}$. Then, by Lemma 4.6, $\pi_\omega(T(\sum_{i\in\check{\mathbf{N}}} s_i x_i))\in\mathscr{H}(\omega)$ for almost all $\omega\in\Omega$. Therefore we are certain that for almost all $\omega\in\Omega$, $\mathscr{H}(\omega)$ is invariant under $\bar{T}(\omega)$. In such a case the restriction of $\bar{T}(\omega)$ to $\mathscr{H}(\omega)$ is denoted by $T(\omega)$. By taking $T(\omega)$ to be the zero operator for $\omega\in\Omega$ in which $\mathscr{H}(\omega)$ is not invariant under $\bar{T}(\omega)$, we obtain a field $\{T(\omega)\}_{\omega\in\Omega}$ of bounded operators on $\mathfrak{H}$, which we denote by $\Phi(T)$.

Lemma 4.13. $\{T(\omega)\}_{\omega\in\Omega}$ is a Borel field of bounded operators on $\mathfrak{H}$.

Proof. It is easy to see that

$$\langle T(\omega)f_i(\omega), f_j(\omega)\rangle=\Phi(\langle Tx_{\check{i}}, x_{\check{j}}\rangle)(\omega)$$

for almost all $\omega\in\Omega$, which implies the desired conclusion by Proposition 4.12. ■

We note that $\Phi(T)$ is determined uniquely up to equivalence, irrespective of our choice of an orthonormal basic sequence $\{x_i\}_{i\in\check{\mathbf{N}}}$ of $\mathscr{H}$ in $V^{(\mathbf{B})}$ in the construction of $\Phi(\mathscr{H})$.

Conversely, given a Borel field $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega},\mathfrak{S})$ of complex Hilbert spaces over Ω and a Borel field $\mathfrak{T}=\{T(\omega)\}_{\omega\in\Omega}$ of bounded operators on $\mathfrak{H}$, we are going to construct a bounded operator $\Psi(\mathfrak{T})$ on $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$. We use the same notation as that in the construction of $\Psi(\mathfrak{H})$ given in the previous subsection. By Theorem 2.4 we can see easily that the function $\Psi(x)\mapsto\Psi(\mathfrak{T}x)$ $(x\in\mathfrak{S})$ naturally induces an operator T on $\mathscr{H}$ in $V^{(\mathbf{B})}$. By

Proposition 4.11 the function $\in\omega\Omega \mapsto \|T(\omega)\|$ is a Borel function, which we denote by f. Then it is easy to see that $\|\Psi(\mathfrak{T}x)\| \leq \Psi(f)\|\Psi(x)\|$ in $V^{(\mathbf{B})}$ for any $x \in \mathfrak{S}$, which implies that T is a bounded operator on $\mathscr{H}$ in $V^{(\mathbf{B})}$. We denote T by $\Psi(\mathfrak{T})$.

Now the following theorem follows readily from the definitions.

Theorem 4.14. For any bounded operator T on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have $T = \Psi(\Phi(T))$, provided we identify $\mathscr{H}$ and $\Psi(\Phi(\mathscr{H}))$ by Theorem 4.9. Conversely, for any Borel field $\mathfrak{T}$ of bounded operators on a Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω, $\Phi(\Psi(\mathfrak{T}))$ and τ are equivalent, provided we identify $\Phi(\Psi(\mathfrak{H}))$ and $\mathfrak{H}$ by Theorem 4.9.

We can naturally define such fundamental operations as addition, multiplication by a complex-valued Borel function, multiplication, and adjoint on Borel fields of bounded operators. Let $\mathfrak{H} = (\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω. Let $\mathfrak{T} = \{T(\omega)\}_{\omega\in\Omega}$ and $\mathfrak{U} = \{U(\omega)\}_{\omega\in\Omega}$ be Borel fields of bounded operators on $\mathfrak{H}$. Let f be a complex-valued Borel function on Ω. We define:

1. $\mathfrak{T} + \mathfrak{U} = \{T(\omega) + U(\omega)\}_{\omega\in\Omega}$.
2. $f\mathfrak{T} = \{f(\omega)T(\omega)\}_{\omega\in\Omega}$.
3. $\mathfrak{T}\mathfrak{U} = \{T(\omega)U(\omega)\}_{\omega\in\Omega}$.
4. $\mathfrak{T}^* = \{T(\omega)^*\}_{\omega\in\Omega}$.

Then it is easy to see the following result.

Proposition 4.15. Let T and U be bounded operators on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Let $r \in \mathbf{C}^{(\mathbf{B})}$. Then:

(a) $\Phi(T+U)$ and $\Phi(T) + \Phi(U)$ are equivalent.
(b) $\Phi(rT)$ and $\Phi(r)\Phi(T)$ are equivalent.
(c) $\Phi(TU)$ and $\Phi(T)\Phi(U)$ are equivalent.
(d) $\Phi(T^*)$ and $\Phi(T)^*$ are equivalent.

To conclude this subsection, we deal with the strong convergence of bounded operators. Let $\mathscr{H}$ be a separable complex Hilbert space in $V^{(\mathbf{B})}$. Let $\{T_i\}_{i\in\check{\mathrm{N}}}$ be a sequence of bounded operators on $\mathscr{H}$ in $V^{(\mathbf{B})}$, which corresponds, by Theorems 2.4 and 4.14, to a sequence $\{\mathfrak{T}_i\}_{i\in\mathrm{N}}$ of Borel fields of bounded operators on $\Phi(\mathscr{H})$, where $\mathfrak{T}_i = \{T_i(\omega)\}_{\omega\in\Omega}$. Then we have the following result.

Theorem 4.16. The sequence $\{T_i\}_{i\in\check{\mathrm{N}}}$ converges strongly to a bounded operator T in $V^{(\mathbf{B})}$ iff (a) $\sup_{i\in N}\|T_i(\omega)\| < +\infty$ for almost all $\omega \in \Omega$, and (b) the sequence $\{T_i(\omega)\}_{i\in\mathrm{N}}$ converges strongly to $T(\omega)$ for almost all $\omega \in \Omega$, where $\Phi(T) = \{T(\omega)\}_{\omega\in\Omega}$.

Proof. We recall (in ZFC) that for a sequence $\{T_i\}_{i\in\mathbf{N}}$ of bounded operators on a complex Hilbert space to converge strongly to a bounded operator S, it is necessary and sufficient that $\sup_{i\in\mathbf{N}}\|S_i\|<+\infty$ and $Sx=\lim_{i\to\infty} S_i x$ for any x of a total subset of the complex Hilbert space, for which the reader is referred, e.g., to Dunford and Schwartz (1958/1963/1971, II.3.6, pp. 60–61). By Proposition 4.11 it is easy to see that $\sup_{i\in\mathbf{N}}\|T_i\|<+\infty$ in $V^{(\mathbf{B})}$ iff $\sup_{i\in\mathbf{N}}\|T_i(\omega)\|<+\infty$ for almost all $\omega\in\Omega$. Using the same notation as in the construction of $\Phi(\mathscr{H})$ given in the previous subsection, we notice, by Theorem 3.4, that $Tx_j=\lim_{i\to\infty} T_i x_j$ for any $j\in\check{\mathbf{N}}$ in $V^{(\mathbf{B})}$ iff $T(\omega)f_j(\omega)=\lim_{i\to\infty} T_i(\omega)f_j(\omega)$ for all $j\in\mathbf{N}$ almost everywhere on Ω. Since $\{x_j\}_{j\in\check{\mathbf{N}}}$ is total in $\mathscr{H}$ in $V^{(\mathbf{B})}$ and $\{f_j(\omega)\}_{j\in\mathbf{N}}$ is total in $\mathscr{H}(\omega)$ for any $\omega\in\Omega$, the desired conclusion follows. ■

To conclude this subsection, we comment that this subsection can be generalized to bounded linear transformations (between different complex Hilbert spaces) without difficulty.

4.3. Von Neumann Algebras

Let $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega},\mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω. A family $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ of von Neumann algebras $\mathscr{A}(\omega)$ on $\mathscr{H}(\omega)$ is called a *Borel field of von Neumann algebras on* $\mathfrak{H}$ if there exists a sequence, called a *generating sequence* of $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$, $\{T_0(\omega)\}_{\omega\in\Omega}$, $\{T_1(\omega)\}_{\omega\in\Omega},\ldots$ of Borel fields of bounded operators on $\mathscr{H}$ such that for almost all $\omega\in\Omega$, $\mathscr{A}(\omega)$ is generated by the sequence $T_0(\omega)$, $T_1(\omega),\ldots$. Two Borel fields $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$, $\{\mathscr{M}(\omega)\}_{\omega\in\Omega}$ of von Neumann algebras on $\mathfrak{H}$ are called *equivalent* if $\mathscr{A}(\omega)=\mathscr{M}(\omega)$ for almost all $\omega\in\Omega$.

Let $\mathscr{A}$ be a von Neumann algebra acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Since $\mathscr{H}$ is separable in $V^{(\mathbf{B})}$, the von Neumann algebra $\mathscr{A}$ is generated by a sequence $\{T_i\}_{i\in\check{\mathbf{N}}}$ of bounded operators in $\mathscr{A}$, for which the reader is referred, e.g., to Bourbaki (1953/1955, Chapter III, § 3, Proposition 6). By Theorems 2.4 and 4.14 the sequence $\{T_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ corresponds externally to a sequence $\{\{T_i(\omega)\}_{\omega\in\Omega}\}_{i\in\mathbf{N}}$ of Borel fields of bounded operators on $\Phi(\mathscr{H})$. Let $\mathscr{A}(\omega)$ be the von Neumann algebra generated by the sequence $\{T_i(\omega)\}_{i\in\mathbf{N}}$ for each $\omega\in\Omega$. It is easy to see that the family $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$, to be denoted by $\Phi(\mathscr{A})$, is a Borel field of von Neumann algebras on $\Phi(\mathscr{H})$, but we need to verify the following result:

Lemma 4.17. $\Phi(\mathscr{A})$ is determined uniquely up to equivalence, irrespective of our choice of the sequence $\{T_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ in the above construction.

Proof. Let $\{\bar{T}_i\}_{i\in\check{\mathbf{N}}}$ be another sequence of bounded operators on $\mathscr{H}$ in $V^{(\mathbf{B})}$ such that it generates $\mathscr{A}$ in $V^{(\mathbf{B})}$. Let $\{\{\bar{T}_i(\omega)\}_{\omega\in\Omega}\}_{i\in\mathbf{N}}$ be the external counterpart of $\{\bar{T}_i\}_{i\in\check{\mathbf{N}}}$ and let $\bar{\mathscr{A}}(\omega)$ be the von Neumann algebra generated by the sequence $\{\bar{T}_i(\omega)\}_{i\in N}$ for any $\omega\in\Omega$. By replacing $\{T_i\}_{i\in\check{N}}$ by $\{T_i/(\|T_i\|+\check{\varepsilon})\}_{i\in\check{N}}$ $(\varepsilon>0)$ if necessary, we can assume without loss of generality that $\{T_i\}_{i\in\check{N}}\subset(\mathscr{A})_1$, where $(\mathscr{A})_1$ is the unit ball of $\mathscr{A}$ endowed with the strong topology. Since $(\mathscr{A})_1$ is metrizable and separable in $V^{(\mathbf{B})}$ (cf. Bourbaki, 1953/1955, Chapter III, § 3, Proposition 6), Kaplansky's density theorem (cf. Kadison and Ringrose, 1983/1986, Theorem 5.3.5) enables us to approximate each T_i by a sequence in the self-adjoint algebra generated by $\{\bar{T}_i\}_{i\in\check{N}}$ in $V^{(\mathbf{B})}$. Therefore, by using Theorem 4.16, we can see that $\mathscr{A}(\omega)\subset\bar{\mathscr{A}}(\omega)$ for almost all $\omega\in\Omega$. By exchanging the roles of $\{T_i\}_{i\in\check{N}}$ and $\{\bar{T}_i\}_{i\in\check{N}}$ in the above discussion, we can see also that $\bar{\mathscr{A}}(\omega)\subset\mathscr{A}(\omega)$ for almost all $\omega\in\Omega$. Therefore $\mathscr{A}(\omega)=\bar{\mathscr{A}}(\omega)$ for almost all $\omega\in\Omega$. ■

Conversely, suppose that we are given a Borel field $\mathfrak{A}=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ of von Neumann algebras on a Borel field $\mathfrak{H}=\{\mathscr{H}(\omega)\}_{\omega\in\Omega}$ of complex Hilbert spaces over Ω. Let $\{\{T_i(\omega)\}_{\omega\in\Omega}\}_{i\in N}$ be a generating sequence of $\mathfrak{A}$. By Theorems 2.4 and 4.14, the generating sequence $\{\{T_i(\omega)\}_{\omega\in\Omega}\}_{i\in N}$ corresponds internally to a sequence $\{T_i\}_{i\in\check{N}}$ in $V^{(\mathbf{B})}$, which generates a von Neumann algebra, to be denoted by $\Psi(\mathfrak{A})$, acting on $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$. The following lemma can be verified in a similar way to Lemma 4.17.

Lemma 4.18. $\Psi(\mathfrak{A})$ is determined uniquely, irrespective of our choice of $\{\{T_i(\omega)\}_{\omega\in\Omega}\}_{i\in N}$ in the above discussion.

Now the following theorem follows from the definitions.

Theorem 4.19. For each von Neumann algebra $\mathscr{A}$ on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have $\Psi(\Phi(\mathscr{A}))=\mathscr{A}$, provided we identify $\Psi(\Phi(\mathscr{H}))$ and $\mathscr{H}$ under Theorem 4.9. Conversely, for each Borel field $\mathfrak{A}$ of von Neumann algebras on a Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω, we have $\Phi(\Psi(\mathfrak{A}))=\mathfrak{A}$ up to equivalence, provided we identify $\Phi(\Psi(\mathfrak{H}))$ and $\mathfrak{H}$ by Theorem 4.9.

4.4. Subspaces

Let $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega},\mathfrak{S})$ be a Borel field of complex Hilbert spces over Ω. A family $\{\mathscr{K}(\omega3\}_{\omega\in\Omega}$ of closed linear subspaces $\mathscr{K}(\omega)\subset\mathscr{H}(\omega)$ is called a *Borel field of subspaces of* $\mathfrak{H}$ if there exists a sequence $\{x_i\}_{i\in N}$ of Borel vector fields such that $\mathscr{K}(\omega)$ is the closed linear span of the sequence $\{x_i(\omega)\}_{i\in N}$. A Borel field $\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ of subspaces of $\mathfrak{H}$ can naturally be made a Borel field $\mathfrak{R}=(\{\mathscr{K}(\omega)\}_{\omega\in\Omega},\mathfrak{S}_{\mathfrak{R}})$ of complex Hilbert spaces over Ω by taking the totality $\mathfrak{S}_{\mathfrak{R}}$ of Borel fields to the totality of Borel fields x of $\mathfrak{H}$ such that $x(\omega)\in\mathscr{K}(\omega)$ for any $\omega\in\Omega$. We call $\mathfrak{R}$ the Borel field of complex Hilbert spaces over Ω *associated with* the Borel field $\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ of subspaces of $\mathfrak{H}$. In the rest of this paper we usually do not distinguish between a Borel field $\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ of subspaces of $\mathfrak{H}$ and its associated Borel field $(\{\mathscr{K}(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_{\mathfrak{R}})$ of complex Hilbert spaces over Ω.

Since our notion of a Borel field of subspaces is not other than Dixmier's (1981, Part II, Chapter 1, Section 7) notion of a measurable field of subspaces adapted simply to our present context, the elementary properties of his notion carry over to our present context with trivial modifications. In particular, we have the following result.

Proposition 4.20. Let $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω, $\mathscr{K}(\omega)$ be a closed linear subspace of $\mathscr{H}(\omega)$ for any $\omega\in\Omega$, and $E(\omega)$ be the projection corresponding to $\mathscr{K}(\omega)$ for any $\omega\in\Omega$. Then for the family $\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ to be a Borel field of subspaces of $\mathfrak{H}$, it is necessary and sufficient that the family $\{E(\omega)\}_{\omega\in\Omega}$ is a Borel field of bounded operators on $\mathfrak{H}$.

Proof. See Dixmier (1981, p. 173). ■

Two Borel fields $\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ and $\{\mathscr{L}(\omega)\}_{\omega\in\Omega}$ of subspaces of $\mathfrak{H}$ are called *equivalent* if $\mathscr{K}(\omega)=\mathscr{L}(\omega)$ for almost all $\omega\in\Omega$.

Let $\mathscr{H}$ be a separable complex Hilbert space in $V^{(\mathbf{B})}$ and $\mathscr{K}$ be a closed linear subspace of $\mathscr{H}$ in $V^{(\mathbf{B})}$. Then $\mathscr{K}$ is a separable complex Hilbert space in $V^{(\mathbf{B})}$. Then we can naturally regard $\Phi(\mathscr{K})$ as a Borel field of subspaces of $\Phi(\mathscr{H})$.

Conversely, given a Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω and a Borel field Ω of subspaces of $\mathfrak{H}$, $\Psi(\mathfrak{R})$ can naturally be regarded as a closed linear subspace of $\Psi(\mathfrak{H})$.

Theorem 4.9 gives at once the following result.

Theorem 4.21. (a) For any closed linear subspace $\mathscr{K}$ of a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have $\Psi(\Phi(\mathscr{K}))=\mathscr{K}$ in $V^{(\mathbf{B})}$, provided we identify $\Psi(\Phi(\mathscr{H}))$ and $\mathscr{H}$ by Theorem 4.9.

(b) For any Borel field $\mathfrak{R}$ of subspaces of a Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω, $\Phi(\Psi(\mathfrak{R}))$ and $\mathfrak{R}$ are equivalent provided we identify $\Phi(\Psi(\mathfrak{H}))$ and $\mathfrak{H}$ by Theorem 4.9.

A Borel field $\{T(\omega)\}_{\omega\in\Omega}$ of bounded operators is called a *Borel field of projections* if $T(\omega)$ is a projection for any $\omega\in\Omega$. If E is a projection on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, then we can assume by Proposition 4.15 that $\Phi(E)$ is a Borel field of projections, which we will do in the rest of this paper. Now we deal with the relationship between Borel fields of subspaces and Borel fields of projections.

Theorem 4.22. Let $\mathscr{H}$ be a separable complex Hilbert space in $V^{(\mathbf{B})}$. Let $\mathscr{K}$ be a closed linear subspace of $\mathscr{H}$ with corresponding projection E in $V^{(\mathbf{B})}$. Let $\Phi(\mathscr{K})=\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ and $\Phi(E)=\{E(\omega)\}_{\omega\in\Omega}$. Then $E(\omega)$ is the projection corresponding to $\mathscr{K}(\omega)$ for almost all $\omega \in \Omega$. In particular, $\Phi(E)$ can always be considered a Borel field of projections of $\Phi(\mathscr{H})$.

Proof. We use the same notation as in the construction of $\Phi(\mathscr{H})$ in Section 4.1. We take $\{x_i\}_{i\in\check{N}}$ in such a manner that $\{x_{2i}\}_{i\in\check{N}}$ is an orthonormal basic sequence of $\mathscr{K}$ in $V^{(\mathbf{B})}$. Then we have in $V^{(\mathbf{B})}$ that $Ex_{2i}=x_{2i}$ for any $i\in\check{N}$, while $Ex_{2i+1}=0$ for any $i\in\check{N}$. Therefore, for almost all $\omega\in\Omega$, $E(\omega)f_{2i}(\omega)=f_{2i}(\omega)$ for any $i\in\mathbf{N}$, while $E(\omega)f_{2i+1}(\omega)=0$, which implies the desired statement at once. ■

Next we would like to deal with orthogonal complements. A similar method to that in the proof of the preceding theorem gives the following result.

Theorem 4.23. For any closed linear subspace $\mathscr{K}$ of a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have that $\Phi(\mathscr{K}^{\perp})$ is equivalent to $\{\mathscr{K}(\omega)^{\perp}\}_{\omega\in\Omega}$, where $\Phi(\mathscr{K})=\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$.

As for the range projection $R(T)$ of a bounded operator T [$R(T)$ is the projection corresponding the minimal closed subspace containing the range of T], we have the following correspondence theorem.

Theorem 4.24. Let T be a bounded operator acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Then $\Phi(R(T))$ is equivalent to $\{R(T(\omega))\}_{\omega\in\Omega}$, where $\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$.

Proof. Let $\{x_i\}_{i\in\check{N}}$ be an orthonormal basic sequence of $\mathscr{H}$ in $V^{(\mathbf{B})}$ with $\{f_i\}_{i\in\mathbf{N}}$ being its corresponding Borel field of orthonormal bases of $\Phi(\mathscr{H})$. Let $\mathscr{K}$ be the closure of the range of T in $V^{(\mathbf{B})}$. Let $\{y_i\}_{i\in\check{N}}$ be the orthonormal basic sequence of $\mathscr{K}$ obtained from $\{Tx_i\}_{i\in\check{N}}$ by the Gram–Schmidt orthogonalization process. More specifically, the sequence $\{y_i\}_{i\in\check{N}}$ is obtained inductively as follows:

(a) If Tx_i belongs to the linear subspace generated by the preceding y_j, then let y_i be the zero vector.
(b) Otherwise, let y_i be the normalization of the vector obtained from Tx_i by subtracting the orthogonal projection of Tx_i onto the linear subspace generated by the preceding y_j.

Let $\{g_i\}_{i\in\mathbf{N}}$ be the Borel field of orthonormal bases of $\Phi(\mathscr{K})$ corresponding to $\{y_i\}_{i\in\check{N}}$. Let $\Phi(\mathscr{K})=\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$. We can see easily that for almost all $\omega\in\Omega$, the Gram–Schmidt orthogonalization applied to the sequence $\{T(\omega)f_i(\omega)\}_{i\in\mathbf{N}}$ gives the sequence $\{g_i(\omega)\}_{\omega\in\Omega}$. Therefore, for almost all $\omega\in\Omega$, $\mathscr{K}(\omega)$ is the closure of the range of $T(\omega)$, which implies the desired result by Theorem 4.22. ■

This theorem gives the following correspondence theorem for the null projection $N(T)$ of a bounded operator T [$N(T)$ is the projection corresponding to the closed linear subspace annihilated by T].

Corollary 4.25. Let T be a bounded operator acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Then $\Phi(N(T))$ is equivalent to $\{N(T(\omega))\}_{\omega\in\Omega}$, where $\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$.

Proof. By Proposition 4.15 and Theorem 4.24, $\Phi(R(T^*))$ is equivalent to $\{R(T(\omega)^*)\}_{\omega\in\Omega}$. Let $\mathscr{L}$ be the closed linear subspace of $\mathscr{H}$ corresponding to the projection $R(T^*)$ in $V^{(\mathbf{B})}$. Let $\Phi(\mathscr{L})=\{\mathscr{L}(\omega)\}_{\omega\in\Omega}$. The projection $N(T)$ corresponds to $\mathscr{L}^{\perp}$ in $V^{(\mathbf{B})}$. By Theorem 4.22, $\mathscr{L}(\omega)$ corresponds to $R(T(\omega)^*)$ for almost all $\omega\in\Omega$, which implies that $N(T(\omega))$ corresponds to $\mathscr{L}(\omega)^{\perp}$ for almost all $\omega\in\Omega$. Therefore $\Phi(N(T))$ and $\{N(T(\omega))\}_{\omega\in\Omega}$ are equivalent by Theorems 4.22 and 4.23. ■

Now we would like to deal with induction and reduction of von Neumann algebras.

Lemma 4.26. Let T be a bounded operator belonging to a von Neumann algebra $\mathscr{A}$ acting on a separable complex Hilbert space $\mathscr{H}$ or to its commutant $\mathscr{A}'$ in $V^{(\mathbf{B})}$. Let $\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$, $\Phi(\mathscr{A})=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$. Then $T(\omega)$ belongs to $\mathscr{A}(\omega)$ or to $\mathscr{A}'(\omega)$ for almost all $\omega\in\Omega$ according as T belongs to $\mathscr{A}$ or to $\mathscr{A}'$ in $V^{(\mathbf{B})}$.

Proof. As in Lemma 4.17, we can assume without loss of generality that $\|T\|\leq\check{1}$ in $V^{(\mathbf{B})}$. We use the same notation as in the construction of $\Phi(\mathscr{A})$ in Section 4.3. Without loss of generality, we can take the sequence $\{T_i\}_{i\in\check{N}}$ generating $\mathscr{A}$ in $V^{(\mathbf{B})}$ to be closed under multiplication and the adjoint operation. We can see that if T belongs to $\mathscr{A}$ in $V^{(\mathbf{B})}$, then a sequence of finite linear combinations of $\{T_i\}_{i\in\check{N}}$ converges strongly to T in $V^{(\mathbf{B})}$, which implies, by Theorem 4.16, that a sequence of finite linear combinations of $\{T_i(\omega)\}_{i\in N}$ converges strongly to $T(\omega)$ for almost all $\omega\in\Omega$, so that $T(\omega)\in\mathscr{A}(\omega)$ for almost all $\omega\in\Omega$. Now we deal with the case that T belongs to $\mathscr{A}'$ in $V^{(\mathbf{B})}$. Let $\{S_i\}_{i\in\check{N}}$ be a sequence of bounded operators in $V^{(\mathbf{B})}$ which generates $\mathscr{A}'$ and which is closed under multiplication and the adjoint operation in $V^{(\mathbf{B})}$. Let $\{\{S_i(\omega)\}_{\omega\in\Omega}\}_{i\in N}$ be the sequence of Borel fields of bounded operators corresponding to $\{S_i\}_{i\in\check{N}}$. Then it is easy to see, by Proposition 4.15, that $S_i(\omega)T_j(\omega)=T_j(\omega)S_i(\omega)$ for all $i,j\in N$ almost everywhere on Ω. Therefore the von Neumann algebra $\mathscr{G}(\omega)$ generated by the sequence $\{S_i(\omega)\}_{i\in N}$ is contained in $\mathscr{A}(\omega)'$. As in the case that T belongs to $\mathscr{A}$ in $V^{(\mathbf{B})}$, we can show that $T(\omega)$ belongs to $\mathscr{G}(\omega)$ for almost all $\omega\in\Omega$. The desired conclusion follows. ■

Theorem 4.27. Let $\mathscr{A}$ be a von Neumann algebra acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Let E be a projection belonging to $\mathscr{A}$ or $\mathscr{A}'$ in $V^{(\mathbf{B})}$. Then $\Phi(\mathscr{A}_E)$ and $\{\mathscr{A}(\omega)_{E(\omega)}\}_{\omega\in\Omega}$ are equivalent, where $\Phi(\mathscr{A})=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ and $\Phi(E)=\{E(\omega)\}_{\omega\in\Omega}$.

Proof. We use the same notation as in the construction of $\Phi(\mathscr{A})$ in Section 4.3. We can take $\{T_i\}_{i\in\check{\mathbf{N}}}$ to be closed under multiplication and the adjoint operation in $V^{(\mathbf{B})}$, which means, by Proposition 4.15, that the sequence $\{T_i(\omega)\}_{i\in\mathbf{N}}$ is closed under multiplication and the adjoint operation for almost all $\omega\in\Omega$. By the second or the third statement of Proposition 1 of Dixmier (1981, Part I, Chapter 2), $\mathscr{A}_E$ is generated by the sequence $\{(T_i)_E\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$, where $(T_i)_E$ is the restriction of the operator ET_i to the range of the projection E. Then, by Proposition 4.15 and Theorem 4.22, the sequence $\{(T_i)_E\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ corresponds externally to

$$\{\{(T_i(\omega))_{E(\omega)}\}_{\omega\in\Omega}\}_{i\in\mathbf{N}}$$

Let $\Phi(\mathscr{A}_E)=\{\mathscr{B}(\omega)\}_{\omega\in\Omega}$, so that $\mathscr{B}(\omega)$ is generated by

$$\{\{(T_i(\omega))_{E(\omega)}\}_{\omega\in\Omega}\}_{i\in\mathbf{N}}$$

for almost all $\omega\in\Omega$. Since $\mathscr{A}(\omega)$ is generated by the sequence $\{T_i(\omega)\}_{i\in\mathbf{N}}$, Proposition 1 of Dixmier (1981, Part I, Chapter 2) shows that $\mathscr{B}(\omega)$ is $\mathscr{A}(\omega)_{E(\omega)}$ for almost all $\omega\in\Omega$. ■

To conclude this subsection, we deal with the central support C_E of a projection E in a von Neumann algebra $\mathscr{A}$ acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$.

Theorem 4.28. $\Phi(C_E)$ is equivalent to $\{C_{E(\omega)}\}_{\omega\in\Omega}$, where

$$\Phi(E)=\{E(\omega)\}_{\omega\in\Omega}$$

Proof. We use the same notation as in the construction of $\Phi(\mathscr{A})$ in Section 4.3. Let $\mathscr{K}$ and $\mathscr{L}$ be the closed linear subspaces of $\mathscr{H}$ corresponding respectively to the projections E and C_E with $\Phi(\mathscr{K})=\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ and $\Phi(\mathscr{L})=\{\mathscr{L}(\omega)\}_{\omega\in\Omega}$. Let $\{y_i\}_{i\in\check{\mathbf{N}}}$ be an orthonormal basic sequence of $\mathscr{K}$ in $V^{(\mathbf{B})}$ corresponding externally to the Borel field $\{g_i\}_{i\in\mathbf{N}}$ of orthonormal bases. By Corollary 1 of Proposition 7 of Dixmier (1981, Part I, Chapter 1), $\mathscr{L}$ is the smallest closed linear subspace of $\mathscr{H}$ containing $\{T_iy_j\}_{(i,j)\in\check{\mathbf{N}}\times\check{\mathbf{N}}}$. By a similar argument to that in Theorem 4.24, we can see that, for almost all $\omega\in\Omega$, $\mathscr{L}(\omega)$ is the smallest closed linear subspace containing

$$\{T_i(\omega)g_j(\omega)\}_{(i,j)\in\mathbf{N}\times\mathbf{N}}$$

which, by Corollary 1 of Proposition 7 of Dixmier (1981, Part I, Chapter 1), corresponds to the projection $C_{E(\omega)}$ almost everywhere on Ω. Then Theorem 4.22 gives the desired conclusion. ■

4.5. Direct Sums and Tensor Products

Let $\mathfrak{H}_1=(\{\mathscr{H}_1(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_1)$ and $\mathfrak{H}_2=(\{\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_2)$ be Borel fields of complex Hilbert spaces over Ω. Then it is easy to see that $(\{\mathscr{H}_1(\omega)\oplus\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ is a Borel field of complex Hilbert spaces over Ω, where $\mathfrak{S}$ consists of all vector fields $\omega\mapsto x_1(\omega)\oplus x_2(\omega)$ with $x_1\in\mathfrak{S}_1$ and $x_2\in\mathfrak{S}_2$. This Borel field of complex Hilbert spaces over Ω is denoted by $\mathfrak{H}_1\oplus\mathfrak{H}_2$ and called the *direct sum* of $\mathfrak{H}_1$ and $\mathfrak{H}_2$.

The following theorem follows directly from the definitions.

Theorem 4.29. For any separable complex Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ in $V^{(\mathbf{B})}$, $\Phi(\mathscr{H}_1\oplus\mathscr{H}_2)$ is equivalent to $\Phi(\mathscr{H}_1)\oplus\Phi(\mathscr{H}_2)$.

Let $\mathfrak{H}_1=(\{\mathscr{H}_1(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_1)$ and $\mathfrak{H}_2=(\{\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_2)$ be Borel fields of complex Hilbert spaces over Ω. Then Dixmier's (1981, Part II, Chapter 1, p. 174) Proposition 10 specialized to our present context goes as follows:

Theorem 4.30. There exists a unique Borel field of complex Hilbert spaces over Ω of the form $\mathfrak{H}=(\{\mathscr{H}_1(\omega)\otimes\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ such that for any Borel vector fields x_1 of $\mathfrak{H}_1$ and x_2 of $\mathfrak{H}_2$, the vector field $\omega\in\Omega\mapsto x_1(\omega)\otimes x_2(\omega)$, denoted usually by $x_1\otimes x_2$, is Borel.

This Borel field of complex Hilbert spaces over Ω is denoted by $\mathfrak{H}_1\otimes\mathfrak{H}_2$ and is called the *tensor product* of $\mathfrak{H}_1$ and $\mathfrak{H}_2$.

Theorem 4.31. For any separable complex Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ in $V^{(\mathbf{B})}$, $\Phi(\mathscr{H}_1\otimes\mathscr{H}_2)$ is equivalent to $\Phi(\mathscr{H}_1)\otimes\Phi(\mathscr{H}_2)$.

Proof. We denote by η a bijective function from N onto $N\times N$, which shall be fixed throughout the rest of this paper. We also denote, for the sake of simplicity, by the same symbol η its internal counterpart in $V^{(\mathbf{B})}$, which is a bijective function from $\check{\mathbf{N}}$ to $\check{\mathbf{N}}\times\check{\mathbf{N}}$ in $V^{(\mathbf{B})}$, so that $\eta(\check{n})=\eta(n)^{\vee}$ in $V^{(\mathbf{B})}$ for any $n\in\mathbf{N}$. We denote by η_1 (η_2, respectively) the function assigning to $n\in\mathbf{N}$ the first (the second, respectively) component of $\eta(n)$. Their internal counterparts are denoted by the same symbols. Let $\{x_i\}_{i\in\check{\mathbf{N}}}$ and $\{y_i\}_{i\in\check{\mathbf{N}}}$ be orthonormal basic sequences of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, with $\{f_i\}_{i\in\mathbf{N}}$ and $\{g_i\}_{i\in\mathbf{N}}$ being their corresponding Borel fields of orthonormal bases in $\Phi(\mathscr{H}_1)$ and $\Phi(\mathscr{H}_2)$, respectively. Then it is easy to see that $\{x_{\eta_1(i)}\otimes y_{\eta_2(i)}\}_{i\in\check{\mathbf{N}}}$ is an orthonormal basic sequence of $\mathscr{H}_1\otimes\mathscr{H}_2$ in $V^{(\mathbf{B})}$, which is supposed to correspond externally to the Borel field $\{h_i\}_{i\in\mathbf{N}}$ of orthonormal bases in $\Phi(\mathscr{H}_1\otimes\mathscr{H}_2)$. Since $\{f_{\eta_1(i)}\otimes g_{\eta_2(i)}\}_{i\in\mathbf{N}}$ is a Borel field of orthonormal bases in $\Phi(\mathscr{H}_1)\otimes\Phi(\mathscr{H}_2)$ and

$$\langle f_{\eta_1(i)}(\omega)\otimes g_{\eta_2(i)}(\omega), f_{\eta_1(j)}(\omega)\otimes g_{\eta_2(j)}(\omega)\rangle=\langle h_i(\omega), h_j(\omega)\rangle$$

almost everywhere on Ω for any $i, j\in N$, we obtain the desired conclusion by Proposition 4.4. ∎

A similar argument gives the following result.

Theorem 4.32. Let T_1 and T_2 be bounded operators acting on separable complex Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, in $V^{(\mathbf{B})}$. Then $\Phi(T_1 \otimes T_2)$ is equivalent to $\Phi(T_1) \otimes (T_2) = \{T_1(\omega) \otimes T_2(\omega)\}_{\omega \in \Omega}$ under the identification of $\Phi(\mathscr{H}_1 \otimes \mathscr{H}_2)$ and $\Phi(\mathscr{H}_1) \otimes \Phi(\mathscr{H}_2)$ by the above theorem, where $\Phi(T_1) = \{T_1(\omega)\}_{\omega \in \Omega}$ and $\Phi(T_2) = \{T_2(\omega)\}_{\omega \in \Omega}$.

Theorem 4.33. For any von Neumann algebras $\mathscr{A}_1, \mathscr{A}_2$ acting on separable complex Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, respectively, in $V^{(\mathbf{B})}$, $\Phi(\mathscr{A}_1 \otimes \mathscr{A}_2)$ is equivalent to $\Phi(\mathscr{A}_1) \otimes \Phi(\mathscr{A}_2) = \{\mathscr{A}_1(\omega) \otimes \mathscr{A}_2(\omega)\}_{\omega \in \Omega}$ under the identification of $\Phi(\mathscr{H}_1) \otimes \Phi(\mathscr{H}_2)$ and $\Phi(\mathscr{H}_1 \otimes \mathscr{H}_2)$ by Theorem 4.31. ■

Proof. This follows readily from Theorem 4.32 of this paper and Proposition 6 of Dixmier (1981, Part I, Chapter 2). ■

Corollary 4.34. Let $\mathfrak{A}_1 = \{\mathscr{A}_1(\omega)\}_{\omega \in \Omega}$ be a Borel field of von Neumann algebras on a Borel field $\mathfrak{H}_1 = (\{\mathscr{H}_1(\omega)\}_{\omega \in \Omega}, \mathfrak{S}_1)$ of complex Hilbert spaces over Ω and $\mathfrak{A}_2 = \{\mathscr{A}_2(\omega)\}_{\omega \in \Omega}$ be a Borel field of von Neumann algebras on another Borel field $\mathfrak{H}_2 = (\{\mathscr{H}_2(\omega)\}_{\omega \in \Omega}, \mathfrak{S}_2)$ of complex Hilbert spaces over Ω. Then $\mathfrak{A}_1 \otimes \mathfrak{A}_2 = \{\mathscr{A}_1(\omega) \otimes \mathscr{A}_2(\omega)\}_{\omega \in \Omega}$ is a Borel field of von Neumann algebras on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

Proof. This follows at once from Theorems 4.19 and 4.33. ■

4.6. Automorphisms

Let $\mathfrak{A}_1 = \{\mathscr{A}_1(\omega)\}_{\omega \in \Omega}$ and $\mathfrak{A}_2 = \{\mathscr{A}_2(\omega)\}_{\omega \in \Omega}$ be Borel fields of von Neumann algebras on Borel fields

$$\mathfrak{H}_1 = (\{\mathscr{H}_1(\omega)\}_{\omega \in \Omega}, \mathfrak{S}_1)$$

and

$$\mathfrak{H}_2 = (\{\mathscr{H}_2(\omega)\}_{\omega \in \Omega}, \mathfrak{S}_2)$$

of complex Hilbert spaces over Ω, respectively. A family $\{\sigma(\omega)\}_{\omega \in \Omega \backslash Z}$ of isomorphisms $\sigma(\omega)$ of $\mathscr{A}_1(\omega)$ onto $\mathscr{A}_2(\omega)$ with Z being a meager Borel set of Ω is called a *Borel field of isomorphisms of* $\mathfrak{A}_1$ *onto* $\mathfrak{A}_2$ if it satisfies the following conditions:

I. For any Borel field $\{S(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}_1$ with $S(\omega) \in \mathscr{A}_1(\omega)$ for any $\omega \in \Omega$, there exists a Borel field $\{T(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}_2$ with $T(\omega) \in \mathscr{A}_2(\omega)$ for any $\omega \in \Omega$ such that $T(\omega) = \sigma(\omega)(S(\omega))$ for any $\omega \in \Omega \backslash Z$.

II. For any Borel field $\{T(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}_2$ with $T(\omega) \in \mathscr{A}_2(\omega)$ for any $\omega \in \Omega$, there exists a Borel field $\{S(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}_1$ with $S(\omega)\in\mathscr{A}_1(\omega)$ for any $\omega\in\Omega$ such that $T(\omega)=\sigma(\omega)(S(\omega))$ for any $\omega\in\Omega\backslash Z$.

Two Borel fields $\{\sigma_1(\omega)\}_{\omega\in\Omega\backslash Z_1}$ and $\{\sigma_2(\omega)\}_{\omega\in\Omega\backslash Z_2}$ of isomorphisms of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ are called *equivalent* if $\sigma_1(\omega)=\sigma_2(\omega)$ for almost all $\omega\in\Omega$. If there exists a Borel field of isomorphisms of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are called *algebraically equivalent*. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ happen to be the same, a Borel field of isomorphisms of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ is called a *Borel field of automorphisms of* $\mathfrak{A}_1=\mathfrak{A}_2$.

Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be von Neumann algebras acting on separable complex Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, in $V^{(\mathbf{B})}$ and σ be an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$ in $V^{(\mathbf{B})}$. We would like to show that σ naturally gives rise to a Borel field $\Phi(\sigma)$ of automorphisms of $\Phi(\mathscr{A})$. By the Corollary of Theorem 3 of Dixmier (1981, Part I, Chapter 4) there exists a von Neumann algebra $\mathscr{B}$ acting on a separable complex Hilbert space $\mathscr{H}_1$ in $V^{(\mathbf{B})}$ such that σ can be written as $\sigma_4\circ\sigma_3\circ\sigma_2^{-1}\circ\sigma_1$, where σ_1 and σ_4 are spatial isomorphisms, and $\sigma_2\colon\mathscr{B}\to\mathscr{B}_E$ and $\sigma_3\colon\mathscr{B}\to\mathscr{B}_F$ are inductions with projections E and F in $\mathscr{B}'$ such that $C_E=C_F=I$ (the identity operator) in $V^{(\mathbf{B})}$.

Let $\Phi(\mathscr{B})=\{\mathscr{B}(\omega)\}_{\omega\in\Omega}$. The method of Section 4.2 used in the construction of $\Phi(T)$ applies to σ_1 and σ_4 to yield Borel fields of isomorphisms $\Phi(\sigma_1)=\{\sigma_1(\omega)\}_{\omega\in\Omega\backslash Z_1}$ and $\Phi(\sigma_4)=\{\sigma_4(\omega)\}_{\omega\in\Omega\backslash Z_4}$. Let

$$\Phi(E)=\{E(\omega)\}_{\omega\in\Omega}$$

and

$$\Phi(F)=\{F(\omega)\}_{\omega\in\Omega}$$

By Theorem 4.28 we are certain that $C_{E(\omega)}=C_{F(\omega)}=I$ for almost all $\omega\in\Omega$, which enables use to define Borel fields of isomorphisms $\{\sigma_2(\omega)\}_{\omega\in\Omega\backslash Z_2}$ and $\{\sigma_3(\omega)\}_{\omega\in\Omega\backslash Z_3}$ such that for all $\omega\in\Omega\backslash Z_2$, $\sigma_2(\omega)$ is the induction of $\mathscr{B}(\omega)$ to $\mathscr{B}(\omega)_{E(\omega)}$, and for all $\omega\in\Omega\backslash Z_3$, $\sigma_3(\omega)$ is the induction of $\mathscr{B}(\omega)$ to $\mathscr{B}(\omega)_{F(\omega)}$. We take $\Phi(\sigma)$ to be

$$\{\sigma_4(\omega)\circ\sigma_3(\omega)\circ\sigma_2(\omega)^{-1}\circ\sigma_1(\omega)\}_{\omega\in\Omega\backslash(Z_1\cup Z_2\cup Z_3\cup Z_4)}.$$

The above consideration has shown the following result.

Proposition 4.35. If von Neumann algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ acting on separable complex Hilbert spaces are algebraically equivalent in $V^{(\mathbf{B})}$, then $\Phi(\mathscr{A}_1)$ and $\Phi(\mathscr{A}_2)$ are algebraically equivalent.

Apparently our definition of $\Phi(\sigma)$ depends on our several extrinsic choices. Fortunately, we can see easily the following result.

Proposition 4.36. For any bounded operator T belonging to $\mathscr{A}_1$ in $V^{(\mathbf{B})}$, letting $\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$, $\Phi(\sigma(T))=\{S(\omega)\}_{\omega\in\Omega}$, and $\Phi(\sigma)=\{\sigma(\omega)\}_{\omega\in\Omega\setminus Z}$, we have $S(\omega)=\sigma(\omega)(T(\omega))$ almost everywhere on Ω.

This gives directly the following result.

Corollary 4.37. $\Phi(\sigma)$ is determined uniquely up to equivalence, irrespective of our several extrinsic choices in the construction of $\Phi(\sigma)$.

Proof. Let $\{T_i\}_{i\in\mathbf{N}}$ be a sequence in the unit ball $(\mathscr{A}_1)_1$ which generates $\mathscr{A}_1$. Let $\{\{T_i(\omega)\}_{\omega\in\Omega}\}_{i\in\mathbf{N}}$ be its corresponding sequence of Borel fields of bounded operators. Let $\Phi(\mathscr{A}_1)=\{\mathscr{A}_1(\omega)\}_{\omega\in\Omega}$. For almost all $\omega\in\Omega$ the sequence $\{T_i(\omega)\}_{i\in\mathbf{N}}$ belongs to $(\mathscr{A}_1(\omega))_1$ and generates $\mathscr{A}_1(\omega)$. By Theorem 3 of Dixmier (1981, Part I, Chapter 4), $\sigma(\omega)$ is weakly continuous on $(\mathscr{A}_1(\omega))_1$ for any $\omega\in\Omega\setminus Z$. Then Proposition 4.36 gives the desired conclusion. ■

Conversely, each Borel field Σ of automorphisms of a Borel field $\mathfrak{A}$ of von Neumann algebras gives naturally an automorphism $\Psi(\Sigma)$ of $\Psi(\mathfrak{A})$ in $V^{(\mathbf{B})}$. We have the following result.

Theorem 4.38. For an isomorphism σ between von Neumann algebras acting on separable complex Hilbert spaces in $V^{(\mathbf{B})}$, we have $\Psi(\Phi(\sigma))=\sigma$ under the identification of Theorem 4.19. Conversely, for any Borel field Σ of isomorphisms between Borel fields of von Neumann algebras, $\Phi(\Psi(\Sigma))$ and Σ are equivalent under the identification of Theorem 4.19.

5. HILBERT ALGEBRAS

Let $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω. A family $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ of full Hilbert algebras $\mathscr{A}(\omega)$ whose Hilbert space completions are $\mathscr{H}(\omega)$ is called a *Borel field of Hilbert algebras on* $\mathfrak{H}$ and $\mathfrak{H}$ is called its *associated* Borel field of complex Hilbert spaces over Ω if it satisfies the following conditions:

I. For any Borel vector field x with $x(\omega)$ in $\mathscr{A}(\omega)$ for all $\omega\in\Omega$, the vector field $x^*: \omega\in\Omega\mapsto x(\omega)^*$ is Borel.

II. For any Borel vector fields x and y with $x(\omega)$ and $y(\omega)$ in $\mathscr{A}(\omega)$ for all $\omega\in\Omega$, the vector field $xy: \omega\in\Omega\mapsto x(\omega)y(\omega)$ is Borel.

III. There exists a fundamental sequence $\{x_i\}_{i\in\mathbf{N}}$ of Borel vector fields such that $x_i(\omega)\in\mathscr{A}(\omega)$ for any $i\in\mathbf{N}$ and any $\omega\in\Omega$.

Our notion of a Borel field of Hilbert algebras is no other than Dixmier's (1981, Part II, Chapter 4) notion of a measurable field of Hilbert algebras adapted simply to our present context, except that we have required all $\mathscr{A}(\omega)$ to be full.

Two Borel fields $\{\mathscr{A}_1(\omega)\}_{\omega\in\Omega}$ and $\{\mathscr{A}_2(\omega)\}_{\omega\in\Omega}$ of Hilbert algebras respectively on Borel fields $(\{\mathscr{H}_1(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_1)$ and $(\{\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_2)$ of complex Hilbert spaces over Ω are called *equivalent* if there exists a meager Borel subset Z of Ω and a (Hilbert space) isomorphism $\varphi_\omega: \mathscr{H}_1(\omega) \to \mathscr{H}_2(\omega)$ for each $\omega\in\Omega\backslash Z$ such that:

I. For every $x\in\mathfrak{S}_1$ there exists $y\in\mathfrak{S}_2$ satisfying $y(\omega)=\varphi_\omega(x(\omega))$ for every $\omega\in\Omega\backslash Z$.

II. For every $y\in\mathfrak{S}_2$ there exists $x\in\mathfrak{S}_1$ satisfying $y(\omega)=\varphi_\omega(x(\omega))$ for every $\omega\in\Omega\backslash Z$.

III. For each $\omega\in\Omega\backslash Zk$, φ_ω, if restricted to $\mathscr{A}_1(\omega)$, induces a bijective correspondence between $\mathscr{A}_1(\omega)$ and $\mathscr{A}_2(\omega)$ preserving the product and $*$ operation.

Let $\mathscr{A}$ be a Hilbert algebra in $V^{(\mathbf{B})}$ whose completion $\mathscr{H}$ in $V^{(\mathbf{B})}$ is assumed to be a separable complex Hilbert space in $V^{(\mathbf{B})}$. Now we would like to show that $\mathscr{A}$ yields naturally a Borel field $\Phi(\mathscr{A})$ of Hilbert algebras whose associated Borel field of complex Hilbert spaces over Ω is equivalent to $\Phi(\mathscr{H})=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$. Since $\mathscr{H}$ is separable in $V^{(\mathbf{B})}$ by assumption, there is a sequence $\{y_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ such that it is total in $\mathscr{A}$ and is closed under product and involution $*$ in $V^{(\mathbf{B})}$. By applying the Gram–Schmidt orthogonalization process to $\{y_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$, we obtain an orthonormal basic sequence $\{x_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$, so that, for any $i, j\in\check{\mathbf{N}}$, $x_i x_j$ can be written as a finite linear combination $\sum_{k\in\check{\mathbf{N}}} r_{ij}^k x_k$, in $V^{(\mathbf{B})}$, where for each $i\in\check{\mathbf{N}}$ and each $j\in\check{\mathbf{N}}$, r_{ij}^k are all zero except for a finite number of k's in $V^{(\mathbf{B})}$, and for any $i\in\check{\mathbf{N}}$, x_i^* can be written as a finite linear combination $\sum_{j\in\check{\mathbf{N}}} s_i^j x_j$, where, for each $i\in\check{\mathbf{N}}$, the s_i^j are all zero except for a finite number of j's in $V^{(\mathbf{B})}$. The sequence $\{x_i\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ gives rise externally to a Borel field $\{f_i\}_{i\in\mathbf{N}}$ of orthonormal bases. For almost all $\omega\in\Omega$ we define $f_i(\omega)f_j(\omega)$ and $f_i(\omega)^*$ as finite linear combinations

$$\sum_{k\in N} \Phi(r_{\check{i}\check{j}}^{\check{k}})(\omega) f_k(\omega) \qquad \text{and} \qquad \sum_{j\in N} \Phi(s_{\check{i}}^{\check{j}})(\omega) f_j(\omega)$$

respectively, for any $i, j\in\mathbf{N}$ so if the product and $*$ operation are extended linearly to the linear subspace $\mathscr{A}_0(\omega)$ generated by $\{f_i(\omega)\}_{i\in\mathbf{N}}$, then $\mathscr{A}_0(\omega)$ is a Hilbert algebra whose completion is $\mathscr{H}(\omega)$, since we have that:

(a) The algebraic equations that the sequence $\{f_i(\omega)\}_{i\in\mathbf{N}}$ should satisfy for $\mathscr{A}_0(\omega)$ to be a Hilbert algebra follow from the equations that the sequence $\{x_i\}_{i\in\check{\mathbf{N}}}$ satisfies in $V^{(\mathbf{B})}$.

(b) It is easy to see that

$$f_i(\omega)f_j(\omega) = U_{x_{\check{i}}}(\omega) f_j(\omega)$$

almost everywhere on Ω, where $U_{x_{\bar{i}}}$ is the bounded operator induced by the left multiplication by $x_{\bar{i}}$ and

$$\Phi(U_{x_{\bar{i}}}) = \{U_{x_{\bar{i}}}(\omega)\}_{\omega\in\Omega}$$

from which, for almost all $\omega\in\Omega$, the continuity of left multiplications in $\mathscr{A}_0(\omega)$ follows.

(c) Since x_k can be written as the limit of a sequence of finite linear combinations of terms in $\{x_i x_j\}_{(i,j)\in\check{\mathbf{N}}\times\check{\mathbf{N}}}$ for any $k\in\check{\mathbf{N}}$ in $V^{(\mathbf{B})}$ by assumption, Theorem 3.4 shows that for almost all $\omega\in\Omega$, $f_k(\omega)$ can be written as the limit of a sequence of finite linear combinations of terms in $\{f_i(\omega)f_j(\omega)\}_{(i,j)\in\mathbf{N}\times\mathbf{N}}$ for any $k\in\mathbf{N}$, from which the totality of $\{f_i(\omega)f_j(\omega)\}_{(i,j)\in\mathbf{N}\times\mathbf{N}}$ in $\mathscr{H}(\omega)$ follows.

For $\omega\in\Omega$ in which this procedure does not work, we can choose arbitrarily a Hilbert algebra $\mathscr{A}_0(\omega)$ whose completion is a separable complex Hilbert space. Now we have a family $\{\mathscr{A}_0(\omega)\}_{\omega\in\Omega}$ of Hilbert algebras, whose completions are denoted by $\mathscr{H}(\omega)$'s. By taking $\mathscr{A}(\omega)$ to be the Hilbert algebra of all bounded elements in $\mathscr{H}(\omega)$ with respect to $\mathscr{A}_0(\omega)$ for each $\omega\in\Omega$, we naturally obtain a Borel family $\Phi(\mathscr{A}) = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ whose associated Borel field of complex Hilbert spaces over Ω is equivalent to $\Phi(\mathscr{H})$. The dependence of $\Phi(\mathscr{A})$ on our choice of $\{y_i\}_{i\in\mathbf{N}}(\mathbf{B})$ is superficial, as we will see.

Lemma 5.1. By replacing $\{y_i\}_{i\in\check{\mathbf{N}}}$ by another sequence $\{y_i^{\#}\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ in the above discussion, we obtain another Borel field $\{\mathscr{A}^{\#}(\omega)\}_{\omega\in\Omega}$ of Hilbert algebras on $\Phi(\mathscr{H})$. Then $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ and $\{\mathscr{A}^{\#}(\omega)\}_{\omega\in\Omega}$ are equivalent.

Proof. We can choose again another sequence $\{y_i^{\#\#}\}_{i\in\check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$ such that $y_{2i}^{\#\#} = y_i$ for any $i\in\check{\mathbf{N}}$, while $y_{2i+1}^{\#\#} = y_i^{\#}$ for $i\in\check{\mathbf{N}}$ in $V^{(\mathbf{B})}$. The sequences $\{y_i\}_{i\in\check{\mathbf{N}}}$, $\{y_i^{\#}\}_{i\in\check{\mathbf{N}}}$, and $\{y_i^{\#\#}\}_{i\in\check{\mathbf{N}}}$ yield families $\{\mathscr{A}_0(\omega)\}_{\omega\in\Omega}$, $\{\mathscr{A}_0^{\#}(\omega)\}_{\omega\in\Omega}$, and $\{\mathscr{A}_0^{\#\#}(\omega)\}_{\omega\in\Omega}$ of Hilbert algebras, which then give rise to Borel fields $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$, $\{\mathscr{A}^{\#}(\omega)\}_{\omega\in\Omega}$, and $\{\mathscr{A}^{\#\#}(\omega)\}_{\omega\in\Omega}$ of Hilbert algebras whose associated Borel fields of complex Hilbert spaces over Ω are all equivalent to $\Phi(\mathscr{H})$. Then we can see easily that for almost all $\omega\in\Omega$, $\mathscr{A}_0(\omega)$ and $\mathscr{A}_0^{\#}(\omega)$ are dense subalgebras of $\mathscr{A}_0^{\#\#}(\omega)$. Therefore, for almost all $\omega\in\Omega$, $\mathscr{A}(\omega) = \mathscr{A}^{\#}(\omega) = \mathscr{A}^{\#\#}(\omega)$, which is the desired conclusion. ■

This lemma justifies our usage of the notation $\Phi(\mathscr{A})$.

The following theorem follows readily from the definitions.

Theorem 5.2. Let $\Phi(\mathscr{A}) = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$. Then $\Phi(U(\mathscr{A}))$ and

$$\{U(\mathscr{A}(\omega))\}_{\omega\in\Omega}$$

are equivalent, where $U(\mathscr{A})$ is the von Neumann algebra left-associated with $\mathscr{A}$ in $V^{(\mathbf{B})}$ and $U(\mathscr{A}(\omega))$ is the von Neumann algebra left-associated with $\mathscr{A}(\omega)$ for each $\omega \in \Omega$.

Proof. Use Proposition 1 of Dixmier (1981, Part I, Chapter 5).

Conversely, given a Borel field $\mathfrak{A} = \{\mathscr{A}(\omega)\}_{\omega \in \Omega}$ of Hilbert algebras whose associated Borel field of complex Hilbert spaces over Ω is $\mathfrak{H} = (\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$, we can naturally construct the Hilbert algebra $\Psi(\mathfrak{A})$ in $V^{(\mathbf{B})}$ whose completion is $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$ such that $\Psi(x)\Psi(y) = \Psi(xy)$ and $\Psi(x)^* = \Psi(x^*)$ in $V^{(\mathbf{B})}$ for any $x, y \in \mathfrak{S}$ with $x(\omega), y(\omega) \in \mathscr{A}(\omega)$ for all $\omega \in \Omega$. Then we have the following result.

Lemma 5.3. $\Psi(\mathfrak{A})$ is full in $V^{(\mathbf{B})}$.

Proof. For each $x \in \mathfrak{S}$, if $\Psi(x)$ is bounded in $V^{(\mathbf{B})}$, then $x(\omega)$ is bounded for almost all $\omega \in \Omega$. Since $\mathscr{A}(\omega)$ is full for any $\omega \in \Omega$, $x(\omega) \in \mathscr{A}(\omega)$ for almost all $\omega \in \Omega$. Thence $\Psi(x) \in \Psi(\mathfrak{A})$ in $V^{(\mathbf{B})}$. ■

Now the following theorem follows from the definitions.

Theorem 5.4. For any full Hilbert algebra $\mathscr{A}$ in $V^{(\mathbf{B})}$ whose completion is a separable complex Hilbert space in $V^{(\mathbf{B})}$, $\Psi(\Phi(\mathscr{A}))$ is isomorphic to $\mathscr{A}$ in $V^{(\mathbf{B})}$. For any Borel field $\mathfrak{A}$ of Hilbert algebras over Ω, $\Phi(\Psi(\mathfrak{A}))$ is equivalent to $\mathfrak{A}$.

6. UNBOUNDED OPERATORS

To begin with, we review the rudiments of Stone's (1951) approach to unbounded operators by characteristic matrices, which reduces a large portion of the theory of unbounded operators to that of bounded operators. We recall that a (not necessarily bounded) operator T on a complex Hilbert space $\mathscr{H}$ is called *closed* if its graph $\Gamma(T)$ is closed in $\mathscr{H} \oplus \mathscr{H}$. The operator T is called *regular* if it is closed and its domain $\mathscr{D}(T)$ is dense in $\mathscr{H}$. If T is a closed operator on $\mathscr{H}$, the projection corresponding to $\Gamma(T)$ can be described by a 2×2 matrix (P_{ij}) of bounded operators P_{ij} on $\mathscr{H}$, which is called the *characteristic matrix* of T. Thus the operator T is the function $P_{11}x_1 + P_{12}x_2 \to P_{21}x_1 + P_{22}x_2$, where x_1 and x_2 are arbitrary elements of $\mathscr{H}$.

The following well-known theorems are very useful.

Theorem 6.1. A 2×2 matrix (P_{ij}) of bounded operators on a complex Hilbert space $\mathscr{H}$ is the characteristic matrix of a regular operator on $\mathscr{H}$ iff it satisfies the following conditions:

1. $P_{ij}^* = P_{ji}$ for $i, j = 1, 2$.
2. $\Sigma_{k=1}^{2} P_{ik}P_{kj} = P_{ij}$ for $i, j = 1, 2$.
3. The null projections $N(P_{11})$ and $N(I - P_{22})$ are the zero operator.

Theorem 6.2. Let T be a regular operator on a complex Hilbert space with its adjoint T^*. Then their characteristic matrices (P_{ij}) and (Q_{ij}) are related by the following identities: (a) $Q_{11}=I-P_{22}$, (b) $Q_{12}=P_{12}$, (c) $Q_{21}=P_{12}$, and (d) $Q_{22}=I-P_{11}$.

Theorem 6.3. Let (P_{ij}) be the characteristic matrix of a regular operator T on a complex Hilbert space. Then we have the following:

1. T is bounded iff $P_{11}>rI$ for some $r>0$.
2. T is self-adjoint iff the operators P_{ij} are self-adjoint and commute with each other.
3. T is normal iff the operators P_{ij} are normal and commute with each other.

Theorem 6.4. Let (P_{ij}) be the characteristic matrix of a regular operator T on a complex Hilbert space. Then T has an inverse iff $I-P_{11}$ has an inverse. In this case the characteristic matrix of T^{-1} is given by (Q_{ij}), where $Q_{11}=P_{22}$, $Q_{12}=P_{21}$, $Q_{21}=P_{12}$, and $Q_{22}=P_{11}$.

The proofs of the above theorems can be found in Nussbaum (1964) or Stone (1951).

Let $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω. A family $\{T(\omega)\}_{\omega\in\Omega}$ of regular operators $T(\omega)$ on $\mathscr{H}(\omega)$ is called a *field of regular operators on* $\mathfrak{H}$. Let $(P_{ij}(\omega))$ be the characteristic matrix of $T(\omega)$ for each $\omega\in\Omega$. Then $\{T(\omega)\}_{\omega\in\Omega}$ is called *Borel* if the fields $\{P_{ij}(\omega)\}_{\omega\in\Omega}$ of bounded operators are Borel $(i,j=1,2)$.

Theorem 6.5. A field $\{T(\omega)\}_{\omega\in\Omega}$ of regular operators is Borel iff the family $\{\Gamma(T(\omega))\}_{\omega\in\Omega}$ is a Borel field of subspaces of $\mathfrak{H}\oplus\mathfrak{H}$, where $\Gamma(T(\omega))$ is the graph of $T(\omega)$ for each $\omega\in\Omega$.

Proof. The proof follows from Proposition 4.20. ■

Two Borel fields S and T of regular operators on $\mathfrak{H}$ with characteristic matrices (P_{ij}) and (Q_{ij}) are called *equivalent* if P_{ij} and Q_{ij} are equivalent $(i,j=1,2)$.

The following theorem is attributed to Nussbaum (1964, Corollary 2).

Theorem 6.6. Let $\{T(\omega)\}_{\omega\in\Omega}$ be a Borel field of regular operators on $\mathfrak{H}$. Then, for any Borel vector field x with $x(\omega)\in\mathscr{D}(T(\omega))$ for all $\omega\in\Omega$, the vector field $\omega\in\Omega\mapsto T(\omega)x(\omega)$ is Borel.

Since a bounded operator is a regular operator, we must verify the consistency of our definition of a Borel field of regular operators with that of a Borel field of bounded operators given in Section 4 in case that all the operators in a given field of regular operators happen to be bounded.

Theorem 6.7. Let $\{T(\omega)\}_{\omega\in\Omega}$ be a field of bounded operators on $\mathfrak{H}$. Then it is Borel as a field of bounded operators iff it is Borel as a field of regular operators.

Proof. The if part follows from Theorem 6.6. To see the only-if part, take a fundamental sequence $\{x_i\}_{i\in\mathrm{N}}$ of Borel vector fields of $\mathfrak{H}$. Then $\Gamma(T(\omega))$ is the closed linear span of the sequence $\{(x_i(\omega), T(\omega)x_i(\omega))\}_{i\in\mathrm{N}}$ for any $\omega\in\Omega$. Then $\{\Gamma(T(\omega))\}_{\omega\in\Omega}$ is a Borel field of subspaces of $\mathfrak{H}\oplus\mathfrak{H}$, which implies, by Theorem 6.5, that T is Borel as a field of regular operators. ■

Let T be a regular operator on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. We would like to construct a Borel field $\Phi(T)$ of regular operators on the Borel field $\Phi(\mathscr{H})=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ of complex Hilbert spaces over Ω. Let (P_{ij}) be the characteristic matrix of T in $V^{(\mathbf{B})}$. Let

$$\Phi(P_{\check{i}\check{j}})=\{P_{ij}(\omega)\}_{\omega\in\Omega}, \qquad i,j=1,2$$

By Proposition 4.15 and Corollary 4.25 we are certain that for almost all $\omega\in\Omega$, the 2×2 matrix $(P_{ij}(\omega))$ is the characteristic matrix of a regular operator $T(\omega)$ on $\mathscr{H}(\omega)$, which determines a Borel field

$$\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$$

up to equivalence.

Conversely, given a Borel field $\mathfrak{T}=\{T(\omega)\}_{\omega\in\Omega}$ of regular operators on a Borel field $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ of complex Hilbert spaces over Ω, we would like to construct a regular operator $\Psi(\mathfrak{T})$ on $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$. Let $(P_{ij}(\omega))$ be the characteristic matrix of $T(\omega)$ for each $\omega\in\Omega$. The family $\{(P_{ij}(\omega))\}_{\omega\in\Omega}$ of 2×2 matrices over Ω determines a 2×2 matrix (P_{ij}) of bounded operators in $V^{(\mathbf{B})}$ such that

$$P_{\check{i}\check{j}}=\Psi(\{P_{ij}(\omega)\}_{\omega\in\Omega} \quad \text{in} \quad V^{(\mathbf{B})} \qquad \text{for any} \quad i,j\in\mathrm{N}$$

By Theorem 4.14, Proposition 4.15, and Corollary 4.25, it is easy to see that the 2×2 matrix (P_{ij}) satisfies the conditions in Theorem 6.1 in $V^{(\mathbf{B})}$, so that it determines a regular operator $\Psi(\mathfrak{T})$ on $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$.

By Theorems 4.14 and 6.1, we have the following result.

Theorem 6.8. For any regular operator T on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have $T=\Psi(\Phi(T))$, provided we identify $\mathscr{H}$ and $\Psi(\Phi(\mathscr{H}))$ by Theorem 4.9. Conversely, for any Borel field $\mathfrak{T}$ of regular operators on a Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω, $\Phi(\Psi(\mathfrak{T}))$ and $\mathfrak{T}$ are equivalent, provided we identify $\Phi(\Psi(\mathfrak{H}))$ and $\mathfrak{H}$ by Theorem 4.9.

It is well known that the adjoint of a regular operator is a regular operator. For any Borel field $\mathfrak{T}=\{T(\omega)\}_{\omega\in\Omega}$ of regular operators on a Borel field $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$ of complex Hilbert spaces over Ω, the family $\mathfrak{T}^*=\{T(\omega)^*\}_{\omega\in\Omega}$ is seen to be a Borel field of regular operators by Theorem 6.2. The same theorem gives directly the following result.

Proposition 6.9. For any regular operator T on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, $\Phi(T^*)$ is equivalent to $\Phi(T)^*$.

Now we would like to deal with the polar decomposition of a regular operator. First of all, we need the following lemma on the range projection of a regular operator.

Lemma 6.10. For any regular operator T on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, let $\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$. Then $\Phi(R(T))$ is equivalent $\{R(T(\omega))\}_{\omega\in\Omega}$.

Proof. Let $\Phi(\mathscr{H})=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$. Obviously the range space of T is that of the bounded operator π_2 restricted to the graph $\Gamma(T)$ of T in $V^{(\mathbf{B})}$, where π_2 is the mapping $(x, y)\in\mathscr{H}\otimes\mathscr{H}\mapsto y\in\mathscr{H}$ in $V^{(\mathbf{B})}$. Similarly, the range space of $T(\omega)$ is that of the bounded operator $\pi_2(\omega)$ restricted to the graph $\Gamma(T(\omega))$ of $T(\omega)$ for each $\omega\in\Omega$, where we use the symbol $\pi_2(\omega)$ to denote the mapping $(x, y)\in\mathscr{H}(\omega)\times\mathscr{H}(\omega)\mapsto y\in\mathscr{H}(\omega)$. Therefore $R(T)=R(\pi_2|\Gamma(T))$ in $V^{(\mathbf{B})}$, while $R(T(\omega))=R(\pi_2(\omega)|\Gamma(T(\omega)))$ for any $\omega\in\Omega$. Since $\Phi(\pi_2)$ is equivalent to $\{\pi_2(\omega)\}_{\omega\in\Omega}$ and $\Phi(\Gamma(T))$ is equivalent to $\{\Gamma(T(\omega))\}_{\omega\in\Omega}$, Theorem 4.24 gives the desired result. ■

We recall that a (possibly unbounded) self-adjoint operator T is called *positive* if $\langle Tx, x\rangle\geq 0$ for any $x\in\mathscr{D}(T)$.

Lemma 6.11. Let T be a positive operator on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ with $\Phi(T)=\{T(\omega)\}_{\omega\in\Omega}$; then $T(\omega)$ is a positive operator on $\mathscr{H}(\omega)$ for almost all $\omega\in\Omega$, where $\Phi(\mathscr{H})=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$.

Proof. By Proposition 6.9 we can see that $T(\omega)$ is self-adjoint for almost all $\omega\in\Omega$. It remains to show that for almost all $\omega\in\Omega$, $\langle T(\omega)x, x\rangle\geq 0$ for all $x\in\mathscr{D}(T(\omega))$. Let $\{(x_i, y_i)\}_{i\in\check{\mathbf{N}}}$ be an orthonormal basic sequence in $\Gamma(T)$ in $V^{(\mathbf{B})}$. Let $\{f_i\}_{i\in\mathbf{N}}$ be the sequence of Borel vector fields corresponding to $\{x_i\}_{i\in\check{\mathbf{N}}}$ and $\{g_i\}_{i\in\mathbf{N}}$ be the sequence of Borel vector fields corresponding to $\{y_i\}_{i\in\check{\mathbf{N}}}$. Then it is easy to see that for almost all $\omega\in\Omega$, the sequence $\{(f_i(\omega), g_i(\omega))\}_{i\in\mathbf{N}}$ is an orthonormal basic sequence of $\Gamma(T(\omega))$. Let $\mathbb{S}$ be the totality of sequences $\{s_i\}_{i\in\mathbf{N}}$ of rational complex numbers s_i, all of which, except for a finite number of them, are zero. It is easy to see that $\mathbb{S}$ is a countable set. Let us take arbitrarily $\{s_i\}_{i\in\mathbf{N}}\in\mathbb{S}$, and let $\{t_i\}_{i\in\check{\mathbf{N}}}$ be its corresponding sequence of rational complex numbers in $V^{(\mathbf{B})}$. Since T is positive in $V^{(\mathbf{B})}$,

$$\left\langle \sum_{i\in\check{N}} t_i y_i, \sum_{i\in\check{N}} t_i x_i \right\rangle = \left\langle T\left(\sum_{i\in\check{N}} t_i x_i\right), \sum_{i\in\check{N}} t_i x_i \right\rangle \geq 0$$

in $V^{(\mathbf{B})}$, which means that for almost all $\omega \in \Omega$,

$$\left\langle T(\omega)\left(\sum_{i\in\mathbf{N}} s_i f_i(\omega)\right), \sum_{i\in\mathbf{N}} s_i f_i(\omega)\right\rangle$$

$$= \left\langle \sum_{i\in\mathbf{N}} s_i g_i(\omega), \sum_{i\in\mathbf{N}} s_i f_i(\omega) \right\rangle \geq 0$$

Thence we are certain that for almost all $\omega \in \Omega$,

$$\left\langle T(\omega)\left(\sum_{i\in\mathbf{N}} s_i f_i(\omega)\right), \sum_{i\in\mathbf{N}} s_i f_i(\omega)\right\rangle \geq 0 \qquad \text{for all} \quad \{s_i\}_{i\in\mathbf{N}} \in \mathbb{S}$$

which gives the desired statement. ■

Theorem 6.12. For any regular operator T on a separable complex Hilbert space $\mathscr{H}$ with polar decomposition $T = UH$ in $V^{(\mathbf{B})}$, if $\Phi(T) = \{T(\omega)\}_{\omega\in\Omega}$, $\Phi(U) = \{U(\omega)\}_{\omega\in\Omega}$, and $\Phi(H) = \{H(\omega)\}_{\omega\in\Omega}$, then $U(\omega)H(\omega)$ is the polar decomposition of $T(\omega)$ for almost all $\omega \in \Omega$.

Proof. Since U is a partial isometry in $V^{(\mathbf{B})}$, $UU^*U = U$ in $V^{(\mathbf{B})}$. Therefore, by Proposition 4.15, $U(\omega)U(\omega)^*U(\omega) = U(\omega)$ for almost all $\omega \in \Omega$, which means that $U(\omega)$ is a partial isometry for almost all $\omega \in \Omega$. Since H is a positive operator, Lemma 6.11 gives that $H(\omega)$ is a positive operator for almost all $\omega \in \Omega$. Since $R(H)N(U) = O$ and $R(H) + N(U) = I$, Proposition 4.15, Corollary 4.25, and Lemma 6.10 give that $R(H(\omega))N(U(\omega)) = O$ and $R(H(\omega)) + N(U(\omega)) = I$ for almost all $\omega \in \Omega$, which means that the initial projection of $U(\omega)$ is the closure of the range of H. Since U is a partial isometry whose initial space contains the range of H, it is easy to see that $\Phi(UH)$ is equivalent to $\{U(\omega)H(\omega)\}_{\omega\in\Omega}$. ■

We conclude this section with the function calculus of a (possibly unbounded) invertible positive self-adjoint operator which will be needed in the next section. We recall that a regular operator T and a bounded operator S on the same complex Hilbert space $\mathscr{H}$ are said to *commute* if $ST \subset TS$. As a special case of Theorem 5 of Stone (1951) we have the following result.

Theorem 6.13. For any regular operator T and a bounded self-adjoint operator S with (P_{ij}) being the characteristic matrix of T, S and T commute iff S commutes with all P_{ij} $(i, j = 1, 2)$.

This gives the following result.

Lemma 6.14. If a regular operator T and a projection E acting on a separable complex Hilbert space $\mathscr{H}$ commute in $V^{(\mathbf{B})}$, then $T(\omega)$ and $E(\omega)$ commute for almost all $\omega \in \Omega$, where $\Phi(T) = \{T(\omega)\}_{\omega \in \Omega}$ and $\Phi(E) = \{E(\omega)\}_{\omega \in \Omega}$.

Proof. This follows from Proposition 4.15 and Theorem 6.13. ■

We recall (see, e.g., Kadison and Ringrose, 1983/1986, p. 351) that an increasing sequence $\{E_i\}_{i \in \mathbf{N}}$ of projections with $\bigvee_{i \in \mathbf{N}} E_i = I$ is called a *bounding sequence* for a regular operator T if T and E_i commute for all $i \in \mathbf{N}$. It is easy to see that if a projection E commutes with a self-adjoint operator T and f is a real-valued continuous function on $\mathbf{R}$, then E commutes with $f(T)$. Hence, if $\{E_i\}_{i \in \mathbf{N}}$ is a bounding sequence for a self-adjoint operator T, then $\{E_i\}_{i \in \mathbf{N}}$ is also a bounding sequence for $f(T)$. We know well that in this case the union of all the subspaces corresponding to the E_i is a core of T, i.e., the minimal closed operator which is an extension of the restriction of T to the union of all the subspaces corresponding to the E_i is T itself, for which the reader is referred to Lemma 5.6.14 of Kadison and Ringrose (1983/1986).

Lemma 6.15. If $\{E_i\}_{i \in \check{\mathbf{N}}}$ is a bounding sequence for a regular operator T acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, then $\{E_i(\omega)\}_{i \in \mathbf{N}}$ is a bounding sequence for $T(\omega)$ for almost all $\omega \in \Omega$, where $\Phi(T) = \{T(\omega)\}_{\omega \in \Omega}$ and $\Phi(E_i) = \{E_i(\omega)\}_{\omega \in \Omega}$ for all $i \in \mathbf{N}$.

Proof. By Lemma 6.14 we are certain that for almost all $\omega \in \Omega$, $T(\omega)$ and $E_i(\omega)$ commute ($i \in \mathbf{N}$). Therefore it suffices to show that $\bigvee_{i \in \mathbf{N}} E_i(\omega) = I$, for almost all $\omega \in \Omega$. Let $\Phi(\mathscr{H}) = (\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$. Let $\{x_i\}_{i \in \check{\mathbf{N}}}$ be an orthonormal basic sequence of $\mathscr{H}$ in $V^{(\mathbf{B})}$ and $\{f_i\}_{i \in \mathbf{N}}$ be its corresponding Borel field of orthonormal bases. Since the sequence $\{E_i x_j\}_{i \in \check{\mathbf{N}}}$ converges strongly to x_j for any $j \in \check{\mathbf{N}}$ in $V^{(\mathbf{B})}$, we are certain by Theorem 3.4 that the sequence $\{E_i(\omega) f_j(\omega)\}_{i \in \mathbf{N}}$ converges strongly to $f_j(\omega)$ for any $j \in \mathbf{N}$ almost everywhere on Ω. Since $\{f_j(\omega)\}_{j \in \mathbf{N}}$ is an orthonormal basic sequence in $\mathscr{H}(\omega)$ for any $\omega \in \Omega$, $\bigvee_{i \in \mathbf{N}} E_i(\omega) = I$ for almost all $\omega \in \Omega$. ■

Theorem 6.16. Let f be a complex-valued continuous function on the positive real line $]0, +\infty[$, which induces naturally a continuous function from the positive half-line of the real numbers to the complex numbers in $V^{(\mathbf{B})}$ denoted by the same symbol f. Let T be a (possibly unbounded) invertible positive operator on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Then $\Phi(f(T)) = \{f(T(\omega))\}_{\omega \in \Omega}$ up to equivalence, where $\Phi(T) = \{T(\omega)\}_{\omega \in \Omega}$.

Proof. We remark, by Theorem 6.4 and Lemma 6.11, that $T(\omega)$ is an invertible positive operator for almost all $\omega \in \Omega$. Our argument is divided into two steps.

1. First of all, we deal with the case that T is bounded in $V^{(\mathbf{B})}$. We define a sequence $\{I_n\}_{n \in \mathbf{N}}$ of open intervals as follows:

(a) $I_0 = I_1 = \phi$.
(b) $I_n =]\frac{1}{n}, n[$ for $n \geq 2$

By using the Stone–Weierstrass theorem, we can choose a sequence $\{f_n\}_{n \in \mathbf{N}}$ of polynomials such that for $n \geq 2$,

$$|f(x) - f_n(x)| \leq \frac{1}{n} \qquad \text{on} \quad I_n$$

Let $\{g_n\}_{n \in \check{\mathbf{N}}}$ be a sequence of polynomials in $V^{(\mathbf{B})}$ corresponding externally to $\{f_n\}_{n \in \mathbf{N}}$. Then $f(T)$ is the uniform limit of the sequence $\{g_n(T)\}_{n \in \check{\mathbf{N}}}$ in $V^{(\mathbf{B})}$, while $f(T(\omega))$ is the uniform limit of the sequence $\{f_n(T(\omega))\}_{n \in \mathbf{N}}$ for almost all $\omega \in \Omega$. Since $\Phi(f_n(T)) = \{f_n(T(\omega))\}_{\omega \in \Omega}$ up to equivalence, Theorem 4.16 applies and the case is completely treated.

2. Now we deal with the general case. Let $\{E_n\}_{n \in \check{\mathbf{N}}}$ be a bounding sequence for T in $V^{(\mathbf{B})}$. Let $\Phi(\mathscr{H}) = (\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$. Then, by Lemma 6.15, $\{E_n(\omega)\}_{n \in \mathbf{N}}$ is a bounding sequence for $T_n(\omega)$ for almost all $\omega \in \Omega$, where $\Phi(E_{\check{n}}) = \{E_n(\omega)\}_{\omega \in \Omega}$ for each $n \in \mathbf{N}$. It is easy to see that $\Phi(T | E_{\check{n}}(\mathscr{H})) = \{T(\omega) | E_n(\mathscr{H}(\omega))\}$ up to equivalence. Then

$$\Gamma(f(T)) = \overline{\bigcup_{n \in \mathbf{N}} (B)\Gamma(f(T) | E_n(\mathscr{H}))} = \overline{\bigcup_{n \in \mathbf{N}} (B)\Gamma(f(T | E_n(\mathscr{H})))}$$

while

$$\Gamma(f(T(\omega))) = \overline{\bigcup_{n \in \mathbf{N}} \Gamma(f(T(\omega)) | E_n(\omega)(\mathscr{H}(\omega)))}$$
$$= \overline{\bigcup_{n \in \mathbf{N}} \Gamma(f(T(\omega) | E_n(\omega)(\mathscr{H}(\omega)))}$$

for almost all $\omega \in \Omega$. By using (1),

$$\Phi(f(T | E_{\check{n}}(\mathscr{H}))) = \{f(T(\omega) | E_n(\omega))\}_{\omega \in \Omega}$$

up to equivalence. Therefore

$$\Phi(f(T)) = \{f(T(\omega))\}_{\omega \in \Omega}$$

up to equivalence. ∎

We need a version of Theorem 6.16 with a parameter in later sections.

Theorem 6.17. Let f be a complex-valued continuous function on $]0, +\infty[\times \mathbf{R}$, which induces naturally a continuous function from the direct product of the positive half-line of the real numbers and the set of all real numbers to the set of all complex numbers in $V^{(\mathbf{B})}$ to be denoted by the same symbol f. Let T be a (possibly unbounded) invertible positive operator on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$. Let t be a real number in $V^{(\mathbf{B})}$, which can be represented by a real-valued Borel function α on Ω. Then

$$\Phi(f(T, t)) = \{f(T(\omega), \alpha(\omega))\}_{\omega \in \Omega}$$

up to equivalence, where $\Phi(T) = \{T(\omega)\}_{\omega \in \Omega}$.

Proof. First, deal with the case that f is restricted to $]0, +\infty[\times [-n, +n]$ for each $n \in \mathbf{N}$, which can be treated as in the proof of Theorem 6.16. Then generalize it to the given f. The details are left to the reader. ■

To conclude this section, we comment that our approach can be generalized to regular linear transformations (i.e., between different complex Hilbert spaces) without difficulty, which we will use in later sections.

7. LEFT HILBERT ALGEBRAS

This section parallels Section 5 a great deal, but since the sharp operation # is not bounded, we must use the theory of unbounded operators developed in the previous section in place of the theory of unbounded operators presented in Section 4. We use freely such standard notations of Takesaki (1970) as S, F, J, Δ, π, $\mathscr{L}(\mathscr{A})$, and $\mathscr{A}'$ for the usual objects associated with $\mathscr{A}$. When we treat a family of left Hilbert algebras such as $\{\mathscr{A}(\omega)\}_{\omega \in \Omega}$ and we must distinguish, e.g., S for different $\mathscr{A}(\omega)$'s, we use such self-explanatory notations as $S(\omega)$.

Let $\mathfrak{H} = (\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$ be a Borel field of complex Hilbert spaces over Ω. A family $\{\mathscr{A}(\omega)\}_{\omega \in \Omega}$ of achieved left Hilbert algebras $\mathscr{A}(\omega)$ whose completions are $\mathscr{H}(\omega)$ is called a *Borel field of left Hilbert algebras on* $\mathfrak{H}$ and $\mathfrak{H}$ is called its *associated* Borel field of complex Hilbert spaces over Ω if it satisfies the following conditions:

I. For any Borel vector fields $\{(\omega)\}_{\omega \in \Omega}$, $\{y(\omega)\}_{\omega \in \Omega}$ with $x(\omega)$ and $y(\omega)$ in $\mathscr{A}(\omega)$ for all $\omega \in \Omega$, the vector field $\{x(\omega)y(\omega)\}_{\omega \in \Omega}$ is Borel.

II. There is a sequence $\{x_i\}_{i \in \mathbf{N}}$ of Borel vector fields such that for almost all $\omega \in \Omega$, $x_i(\omega) \in \mathscr{A}(\omega)$ and the sequence $\{(x_i(\omega), x_i(\omega)^{\#})\}_{i \in \mathbf{N}}$ is total in $\Gamma(S(\omega))$, where $S(\omega)$ is the usual closed operator associated with the sharp operation.

Borel fields of right Hilbert algebras are defined similarly and their theory can be developed parallel with the theory of left Hilbert algebras.

Proposition 7.1. In the above definition of a Borel field of left Hilbert algebras we have that the field $\{S(\omega)\}_{\omega\in\Omega}$ of regular operators is Borel.

Proof. This follows from Proposition 4.20. ■

Proposition 7.2. Let $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ be a Borel field of left Hilbert algebras on $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$. Then for any Borel vector field $\{x(\omega)\}_{\omega\in\Omega}$ with $x(\omega)$ in $\mathscr{A}(\omega)$ for all $\omega\in\Omega$, the vector field $\{x(\omega)^{\#}\}_{\omega\in\Omega}$ is Borel.

Proof. This follows from Theorem 6.6 and Proposition 7.1. ■

Two Borel fields $\{\mathscr{A}_1(\omega)\}_{\omega\in\Omega}$ and $\{\mathscr{A}_2(\omega)\}_{\omega\in\Omega}$ of left Hilbert algebras on Borel fields $(\{\mathscr{H}_1(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_1)$ and $(\{\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_2)$ of complex Hilbert spaces over Ω are called *equivalent* if there exist a meager Borel subset Z of Ω and a (Hilbert space) isomorphism $\varphi_\omega: \mathscr{H}_1(\omega)\to\mathscr{H}_2(\omega)$ for each $\omega\in\Omega\backslash Z$ such that:

I. For every $x\in\mathfrak{S}_1$ there exists $y\in\mathfrak{S}_2$ satisfying $y(\omega)=\varphi_\omega(x(\omega))$ for any $\omega\in\Omega\backslash Z$.

II. For every $y\in\mathfrak{S}_2$ there exists $x\in\mathfrak{S}_1$ satisfying $y(\omega)=\varphi_\omega(x(\omega))$ for any $\omega\in\Omega\backslash Z$.

III. For each $\omega\in\Omega\backslash Z$, φ_ω, if restricted to $\mathscr{A}_1(\omega)$, induces a bijective correspondence between $\mathscr{A}_1(\omega)$ and $\mathscr{A}_2(\omega)$ preserving the product and the sharp operation $\#$.

Let $\mathscr{H}$ be a separable complex Hilbert space in $V^{(\mathbf{B})}$ and $\mathscr{A}$ be a left Hilbert algebra in $V^{(\mathbf{B})}$ whose completion is $\mathscr{H}$ in $V^{(\mathbf{B})}$. We would like to construct a Borel field $\Phi(\mathscr{A})$ of left Hilbert algebras over Ω. Let $\{y_i\}_{i\in\check{N}}$ be a sequence of elements of $\mathscr{A}$ in $V^{(\mathbf{B})}$ such that it is closed under the product and the sharp operation $\#$ and it is dense in $\mathscr{D}^{\#}$ in $V^{(\mathbf{B})}$. Then the construction of $\Phi(\mathscr{A})$ in Section 5 where $\mathscr{A}$ is a Hilbert algebra in $V^{(\mathbf{B})}$ carries over literally up to the production of $\{\mathscr{A}_0(\omega)\}_{\omega\in\Omega}$ except that every occurrence of the star operation $*$ should be replaced by the sharp operation $\#$. Let $\mathscr{A}(\omega)=\mathscr{A}_0(\omega)''$ for any $\omega\in\Omega$. It is easy to see that the family $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ is a Borel field of left Hilbert algebras whose associated Borel field of complex Hilbert spaces over Ω is equivalent to $\Phi(\mathscr{H})$. We denote $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ by $\Phi(\mathscr{A})$. By Lemma 5.2 of Takesaki (1970) of Proposition 2.24 of Takesaki (1983, Chapter 1), Lemma 5.1 carries over with due modifications. Therefore $\Phi(\mathscr{A})$ is determined uniquely up to equivalence.

Conversely, let us suppose that we are given a Borel field $\mathfrak{A}=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ of left Hilbert algebras whose associated Borel field of complex Hilbert spaces over Ω is $\mathfrak{H}=(\{\mathscr{H}(\omega)\}_{\omega\in\Omega}, \mathfrak{S})$. Then we can naturally construct the left Hilbert algebra $\Psi(\mathscr{A})$ in $V^{(\mathbf{B})}$ whose completion is $\Psi(\mathfrak{H})$ in $V^{(\mathbf{B})}$ such that $\Psi(x)\Psi(y)=\Psi(xy)$ and $\Psi(x)^{\#}=\Psi(x^{\#})$ in $V^{(\mathbf{B})}$ for any $x, y \in \mathfrak{S}$ with $x(\omega), y(\omega) \in \mathscr{A}(\omega)$ for all $\omega \in \Omega$. By replacing "full" by "achieved," Lemma 5.3 carries over.

The following theorem follows from the definitions.

Theorem 7.3. (a) For any achieved left Hilbert algebra $\mathscr{A}$ in $V^{(\mathbf{B})}$ whose Hilbert space completion $\mathscr{H}$ is separable, we have $\Psi(\Phi(\mathscr{A}))=\mathscr{A}$, provided we identify $\mathscr{H}$ and $\Psi(\Phi(\mathscr{H}))$ by Theorem 4.9.

(b) For any Borel field $\mathfrak{A}$ of left Hilbert algebras on a Borel field $\mathfrak{H}$ of complex Hilbert spaces over Ω, $\Phi(\Psi(\mathfrak{A}))$ and $\mathfrak{A}$ are equivalent, provided we identify $\Phi(\Psi(\mathfrak{H}))$ and $\mathfrak{H}$ by Theorem 4.9.

Theorem 7.4. Let $\mathscr{A}$ be a left Hilbert algebra in $V^{(\mathbf{B})}$ whose Hilbert space completion $\mathscr{H}$ is separable. Then $\Phi(\mathscr{A}')=\{\mathscr{A}(\omega)'\}_{\omega\in\Omega}$ up to equivalence, where $\Phi(\mathscr{A})=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$.

Proof. Let $\Phi(\mathscr{A}')=\{\mathscr{B}(\omega)\}_{\omega\in\Omega}$. It is easy to see that $\mathscr{B}(\omega)\subset\mathscr{A}'(\omega)$ for almost all $\omega\in\Omega$. Since $\beta(\omega)$ is dense in $\mathscr{A}'(\omega)$ with respect to the norm topology in the complex Hilbert space induced by the flat operation for almost all $\omega\in\Omega$ and $\mathscr{B}(\omega)=\mathscr{B}(\omega)''$ by definition, the desired conclusion follows from Lemma 5.2 of Takesaki (1970). ■

Theorem 7.5. For any left Hilbert algebra $\mathscr{A}$ in $V^{(\mathbf{B})}$ whose Hilbert space completion is separable, $\Phi(\mathscr{L}(\mathscr{A}))=\{\mathscr{L}(\mathscr{A}(\omega))\}_{\omega\in\Omega}$ up to equivalence, where $\Phi(\mathscr{A})=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$.

Proof. It suffices to recall (in ZFC) that for any left Hilbert algebra $\mathscr{A}$ and any subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ such that $\mathscr{N}$ is dense in $\mathscr{A}$ with respect to the norm topology in $\mathscr{D}^{\#}$, we have $\mathscr{L}(\mathscr{M})=\mathscr{L}(\mathscr{N})$, for which see Theorem 3.1 and Lemma 5.2 of Takesaki (1983). ■

Theorem 7.6. For any left Hilbert algebra $\mathscr{A}$ in $V^{(\mathbf{B})}$ whose Hilbert space completion is separable, we have the following:

1. $\Phi(J)=\{J(\omega)\}_{\omega\in\Omega}$ and $\Phi(\Delta)=\{\Delta(\omega)\}_{\omega\in\Omega}$ up to equivalence.
2. Let t be a real number in $V^{(\mathbf{B})}$, which can be represented by a real-valued Borel function f on Ω. Then $\Phi(\sigma_t)=\{\sigma_{f(\omega)}(\omega)\}_{\omega\in\Omega}$ up to equivalence.

Proof. (1) Since $\Phi(S)=\{S(\omega)\}_{\omega\in\Omega}$ up to equivalence and J and $\Delta^{1/2}$ [$J(\omega)$ and $\Delta(\omega)^{1/2}$] can be obtained as the polar decomposition of S [of $S(\omega)$], we have, by Theorem 6.12, that $\Phi(J)=\{J(\omega)\}_{\omega\in\Omega}$ and $\Phi(\Delta^{1/2})=\{\Delta(\omega)^{1/2}\}_{\omega\in\Omega}$ up to equivalence. Therefore $\Phi(\Delta)=\{\Delta(\omega)\}_{\omega\in\Omega}$ up to equivalence by Theorem 6.16.

(2) This follows from Part 1 and Theorem 6.17. ■

8. THE COMMUTATION THEOREM

The powerful machinery of left Hilbert algebras is available in our framework; we can establish the following commutation theorem rather easily, just as Tomita–Takesaki theory has settled the commutation theorem for tensor products of von Neumann algebras.

Theorem 8.1. For any von Neumann algebra $\mathscr{M}$ acting on a separable complex Hilbert space in $V^{(\mathbf{B})}$, we have $\Phi(\mathscr{M}')=\{\mathscr{M}(\omega)'\}_{\omega\in\Omega}$ up to equivalence, where $\Phi(\mathscr{M})=\{\mathscr{M}(\omega)\}_{\omega\in\Omega}$.

Before entering into the proof, the reader should understand that it is easy to show that if $\Phi(\mathscr{M}')=\{\mathscr{N}(\omega)\}_{\omega\in\Omega}$, then $\mathscr{N}(\omega)\subset\mathscr{M}(\omega)'$ for almost all $\omega\in\Omega$. The reverse inclusion is much harder to show, for which our machinery plays a decisive role.

Proof of the Theorem. We know in the theory of left Hilbert algebras that each von Neumann algebra is isomorphic to the left von Neumann algebra of a left Hilbert algebra, for which the reader is referred, e.g., to Kadison and Ringrose (1983/1986, § 9.2) or Takesaki (1983, Chapter 2). We know also in theory of von Neumann algebras that an isomorphism between von Neumann algebras can be represented as the composition of an amplication, an induction, and a spatial isomorphism, for which the reader is referred to Dixmier (1981, Part I, Chapter 4, Theorem 3, pp. 61–62). By checking their proofs, we can see easily that these theorems still hold under our separability context. Therefore the von Neumann algebra $\mathscr{M}$ in $V^{(\mathbf{B})}$ can be supposed, without loss of generality, to be of the form $(\mathscr{L}(\mathscr{A})\otimes\mathscr{C}_{\mathscr{H}_2})_E$, where $\mathscr{A}$ is a left Hilbert algebra whose Hilbert space completion $\mathscr{H}_1$ is separable, $\mathscr{L}(\mathscr{A})$ is the left von Neumann algebra of $\mathscr{A}$, $\mathscr{C}_{\mathscr{H}_2}$ is the von Neumann algebra of scalar operators on a separable complex Hilbert space $\mathscr{H}_2$, and E is a projection belonging to the commutant of the von Neumann algebra $\mathscr{L}(\mathscr{A})\otimes\mathscr{C}_{\mathscr{H}_2}$. Therefore, by using Theorems 4.27, 4.33, 7.4, and 7.5, we have

$$\begin{aligned}\Phi(\mathscr{M}') &= \Phi((\mathscr{L}(\mathscr{A})\otimes\mathscr{C}_{\mathscr{H}_2})'_E)\\ &= \Phi((\mathscr{L}(\mathscr{A})'\otimes L(\mathscr{H}_2))_E)\\ &= \Phi((\mathscr{L}(\mathscr{A}')\otimes L(\mathscr{H}_2))_E)\\ &= \{(\mathscr{L}(\mathscr{A}(\omega)')\otimes L(\mathscr{H}_2(\omega)))_{E(\omega)}\}_{\omega\in\Omega}\\ &= \{(\mathscr{L}(\mathscr{A}(\omega))\otimes\mathscr{C}_{\mathscr{H}_2(\omega)})'_{E(\omega)}\}_{\omega\in\Omega}\\ &= \{\mathscr{M}(\omega)'\}_{\omega\in\Omega}\end{aligned}$$

where the above equalities should be understood as "up to equivalence" or an isomorphism case by case, $\Phi(\mathscr{H}_2)=(\{\mathscr{H}_2(\omega)\}_{\omega\in\Omega}, \mathfrak{S}_2)$, and $L(\mathscr{H}_2)$ [$L(\mathscr{H}_2(\omega))$, respectively] denotes the set of all bounded operators on $\mathscr{H}_2$ [$\mathscr{H}_2(\omega)$, respectively]. ■

This theorem gives the following intersection theorem.

Theorem 8.2. For any von Neumann algebras $\mathscr{M}$, $\mathscr{N}$ acting on the same separable complex Hilbert space in $V^{(\mathbf{B})}$,

$$\Phi(\mathscr{M}\cap\mathscr{N})=\{\mathscr{M}(\omega)\cap\mathscr{N}(\omega)\}_{\omega\in\Omega}$$

up to equivalence, where $\Phi(\mathscr{M})=\{\mathscr{M}(\omega)\}_{\omega\in\Omega}$ and $\Phi(\mathscr{N})=\{\mathscr{N}(\omega)\}_{\omega\in\Omega}$.

Proof. By using Theorem 8.1, we have

$$\begin{aligned}\Phi(\mathscr{M}\cap\mathscr{N})&=\Phi((\mathscr{M}'\cup\mathscr{N}')')\\&=\{(\mathscr{M}(\omega)'\cup\mathscr{N}(\omega)')'\}_{\omega\in\Omega}\\&=\{\mathscr{M}(\omega)\cap\mathscr{N}(\omega)\}_{\omega\in\Omega}\end{aligned}$$

where the equality should be understood as "up to equivalence." ■

The above two theorems give the following result.

Theorem 8.3. For any von Neumann algebra $\mathscr{M}$ acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, we have $\Phi(\mathscr{Z}(\mathscr{M}))=\{\mathscr{Z}(\mathscr{M}(\omega))\}_{\omega\in\Omega}$ up to equivalence, where $\Phi(\mathscr{M})=\{\mathscr{M}(\omega)\}_{\omega\in\Omega}$ and $\mathscr{Z}(\mathscr{M})$ [$\mathscr{Z}(\mathscr{M}(\omega))$] denotes the center of $\mathscr{M}$ [of $\mathscr{M}(\omega)$].

Proof. Since $\mathscr{Z}(\mathscr{M})=\mathscr{M}\cap\mathscr{M}'$ in $V^{(\mathbf{B})}$ and $\mathscr{Z}(\mathscr{M}(\omega))=\mathscr{M}(\omega)\cap\mathscr{M}(\omega)'$ for any $\omega\in\Omega$, we have, by Theorems 8.1 and 8.2, that

$$\begin{aligned}\Phi(\mathscr{Z}(\mathscr{M}))&=\Phi(\mathscr{M}\cap\mathscr{M}')\\&=\{\mathscr{M}(\omega)\cap\mathscr{M}(\omega)'\}_{\omega\in\Omega}\\&=\Phi(\mathscr{Z}(\mathscr{M}(\omega)))\end{aligned}$$

where the equality should be understood as "up to equivalence." ■

This theorem gives the following factor theorem.

Theorem 8.4. For any von Neumann algebra $\mathscr{M}$ acting on a separable complex Hilbert space, $\mathscr{M}$ is a factor in $V^{(\mathbf{B})}$ iff $\mathscr{M}(\omega)$ is a factor for almost all $\omega\in\Omega$.

Proof. This follows from Theorems 4.19 and 8.3. ■

9. THE SEMIFINITENESS THEOREM

We owe much of the basic idea of this section to Kallman (1971) and Lance (1975).

Theorem 9.1. Let $\mathscr{M}$ be a factor acting on a separable complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ and σ be an automorphism of $\mathscr{M}$ in $V^{(\mathbf{B})}$. Let $\Phi(\mathscr{M}) = \{\mathscr{M}(\omega)\}_{\omega \in \Omega}$ and $\Phi(\sigma) = \{\sigma(\omega)\}_{\omega \in \Omega}$. If $\sigma(\omega)$ is inner for almost all $\omega \in \Omega$, then σ is inner in $V^{(\mathbf{B})}$.

Proof. Let $\Phi(\mathscr{H}) = (\{\mathscr{H}(\omega)\}_{\omega \in \Omega}, \mathfrak{S})$. We use the techniques of Takesaki (1969). By Theorem 8.4, $\mathscr{M}(\omega)$ is a factor for almost all $\omega \in \Omega$. We proceed as in the proof of Proposition 3.2 of Lance (1975) to get a Borel field $\mathfrak{U} = \{U(\omega)\}_{\omega \in \Omega}$ of bounded operators on $\mathfrak{H}$ such that all $U(\omega)$ are unitary operators inducing $\sigma(\omega)$. It is easy to see that $\Psi(\mathfrak{U})$ induces the automorphism σ in $V^{(\mathbf{B})}$. ∎

Now we can establish the following semifiniteness theorem rather easily by using Kallman's (1971) main theorem on inner automorphisms of von Neumann algebras.

Theorem 9.2. For a factor $\mathscr{M}$ acting on a separable complex Hilbert space in $V^{(\mathbf{B})}$, $\mathscr{M}$ is semifinite iff $\mathscr{M}(\omega)$ is semifinite for almost all $\omega \in \Omega$, where $\Phi(\mathscr{M}) = \{\mathscr{M}(\omega)\}_{\omega \in \Omega}$.

Proof. (1) First we deal with the if part. As in the proof of Theorem 8.1, in $V^{(\mathbf{B})}$ the von Neumann algebra $\mathscr{M}$ can be assumed to be of the form $\mathscr{L}(\mathscr{A})$ for some left Hilbert algebra $\mathscr{A}$ whose Hilbert space completion is separable. Then $\mathscr{M}(\omega)$ is of the form $\mathscr{L}(\mathscr{A}(\omega))$ for almost all $\omega \in \Omega$ with $\Phi(\mathscr{A}) = \{\mathscr{A}(\omega)\}$, which follows from Theorem 7.5. To prove that $\mathscr{M}$ is semifinite in $V^{(\mathbf{B})}$, it suffices, by Takesaki (1970, § 14), to show that the associated modular automorphism group $\{\sigma_t\}_{t \in \mathbf{R}}(\mathbf{B})$ is inner, for which it then suffices, by Kallman's (1971) main theorem, to show that σ_t is inner for each $t \in \mathbf{R}^{(\mathbf{B})}$ in $V^{(\mathbf{B})}$. Let t be a real number in $V^{(\mathbf{B})}$, which can be represented by a real-valued Borel function f on Ω. Then, by Theorem 7.6, we have $\Phi(\sigma_t) = \{\sigma_{f(\omega)}(\omega)\}_{\omega \in \Omega}$. By Takesaki (1970, § 14) again, $\sigma_{f(\omega)}(\omega)$ is inner for almost all $\omega \in \Omega$, which implies, by Theorem 9.1, that σ_t is inner in $V^{(\mathbf{B})}$. This completes the proof of the if part.

(2) The only-if part follows from the succeeding simple observation. We recall (cf. Dixmier, 1981, Part I, Chapter 6, § 7, Corollary of Proposition 9) that a von Neumann algebra is semifinite iff it is isomorphic to the left von Neumann algebra of a Hilbert algebra. Since $\mathscr{M}$ is semifinite by assumption, $\mathscr{M}$ can be supposed to be the von Neumann algebra $U(\mathscr{A})$ left-associated with a Hilbert algebra $\mathscr{A}$. Then $\mathscr{M}(\omega)$ is of the form $U(\mathscr{A}(\omega))$

for almost all $\omega \in \Omega$ with $\Phi(\mathscr{A}) = \{\mathscr{A}(\omega)\}_{\omega \in \Omega}$, as follows from Theorem 5.2. This implies that $\mathscr{M}(\omega)$ is semifinite for almost all $\omega \in \Omega$. ■

We notice that the only-if part in the above theorem still holds without the factor assumption. Indeed the proof for the only-if part is elementary. A similar elementary approach to the if part would break down, for although there exists a Hilbert algebra whose left von Neumann algebra is $\mathscr{M}(\omega)$ for almost all $\omega \in \Omega$, there exists no elementary method to combine these Hilbert algebras into a Borel field of Hilbert algebras. This is why our machinery was more than an extravagant toy for pedants in the proof of the theorem.

10. TYPE CORRESPONDENCE THEOREMS

Let $\mathscr{M}$ be a semifinite von Neumann algebra acting on a separable complex Hilbert space in $V^{(\mathbf{B})}$, which shall be fixed throughout this section. Let $\Phi(\mathscr{M}) = \{\mathscr{M}(\omega)\}_{\omega \in \Omega}$. First of all, we deal with the semifinite case.

Theorem 10.1. If $\mathscr{M}$ is discrete in $V^{(\mathbf{B})}$, then $\mathscr{M}(\omega)$ is discrete for almost all $\omega \in \Omega$.

Proof. By Dixmier (1981, Part I, Chapter 8, Theorem 1), there exists an Abelian projection E of $\mathscr{M}$ with central support I in $V^{(\mathbf{B})}$. Let $\Phi(E) = \{E(\omega)\}_{\omega \in \Omega}$. By Proposition 4.15 and Theorem 4.28, $E(\omega)$ is an Abelian projection of $\mathscr{M}(\omega)$ with central support I for almost all $\omega \in \Omega$. Therefore, by Dixmier (1981, Part I, Chapter 8, Theorem 1) again, the desired result follows. ■

Theorem 10.2. If $\mathscr{M}$ is properly infinite, then $\mathscr{M}(\omega)$ is properly infinite for almost all $\omega \in \Omega$.

Proof. By the halving theorem (see, e.g., Kadison and Ringrose, 1981/1986, Lemma 6.3.3), there exists a projection E of $\mathscr{M}$ in $V^{(\mathbf{B})}$ such that $I \sim E \sim I - E$. Let $\Phi(E) = \{E(\omega)\}_{\omega \in \Omega}$. Then $I \sim E(\omega) \sim I - E(\omega)$ for almost all $\omega \in \Omega$. Therefore $\mathscr{M}(\omega)$ is properly infinite for almost all $\omega \in \Omega$. ■

Now we would like to show the following result.

Theorem 10.3. If $\mathscr{M}$ is finite in $V^{(\mathbf{B})}$, then $\mathscr{M}(\omega)$ is finite for almost all $\omega \in \Omega$.

Proof. Let $\mathscr{A}$ be a full Hilbert algebra in $V^{(\mathbf{B})}$ whose Hilbert space completion is separable in $V^{(\mathbf{B})}$ and whose left-associated von Neumann algebra $U(\mathscr{A})$ is algebraically isomorphic to $\mathscr{M}$ in $V^{(\mathbf{B})}$. Using the correspondence between Hilbert algebras and faithful semifinite normal traces (see, e.g., Dixmier, 1981, Part I, Chapter 6, § 2), we let φ be the faithful semifinite normal trace on $\mathscr{M}^+$ corresponding to the Hilbert algebra $\mathscr{A}$ in

$V^{(\mathbf{B})}$. By Dixmier (1981, Part I, Chapter 6, Proposition 9), we can assume that φ is finite in $V^{(\mathbf{B})}$. Since $\varphi(I)$ is finite in $V^{(\mathbf{B})}$, there exists an element a of $\mathscr{A}$ in $V^{(\mathbf{B})}$ such that $I=U_a$ and $\varphi(I)=\langle a, a\rangle$ in $V^{(\mathbf{B})}$, where U_a is the bounded linear operator induced by the left multiplication of a on $\mathscr{A}$. Let $\Phi(\mathscr{A})=\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ and $\Phi(a)=\{a(\omega)\}_{\omega\in\Omega}$. By Theorem 5.2 we can identify $\mathscr{M}(\omega)$ and $U(\mathscr{A}(\omega))$ for almost all $\omega\in\Omega$. Let φ_ω be the faithful semifinite normal trace on $\mathscr{M}(\omega)^+$ corresponding to $\mathscr{A}(\omega)$ for almost all $\omega\in\Omega$. Then it is easy to see that $\varphi_\omega(I)=\langle a(\omega), a(\omega)\rangle$ for almost all $\omega\in\Omega$, which means that $\varphi_\omega(I)$ is finite for almost all $\omega\in\Omega$. Therefore $\mathscr{M}(\omega)$ is finite for almost all $\omega\in\Omega$. ■

Corollary 10.4. Let E be a projection of $\mathscr{M}$ in $V^{(\mathbf{B})}$ with $\Phi(E)=\{E(\omega)\}_{\omega\in\Omega}$. If E is a finite projection in $V^{(\mathbf{B})}$, then $E(\omega)$ is finite for almost all $\omega\in\Omega$.

Proof. Since $\Phi(\mathscr{M}_E)=\{\mathscr{M}(\omega)_{E(\omega)}\}_{\omega\in\Omega}$ up to equivalence by Theorem 4.27, the desired result follows from Theorem 10.3.

Theorem 10.5. If $\mathscr{M}$ is continuous in $V^{(\mathbf{B})}$, then $\mathscr{M}(\omega)$ is continuous for almost all $\omega\in\Omega$.

Proof. By Dixmier (1981, Part III, Chapter 2, Corollary 4 of Proposition 7), there exists a decreasing sequence $\{E_n\}_{n\in\check{\mathrm{N}}}$ of finite projections of $\mathscr{M}$ with central support I in $V^{(\mathbf{B})}$ such that $E_n \sim E_n - E_{n+1}$ for every $n\in\check{\mathrm{N}}$ in $V^{(\mathbf{B})}$. Let $\Phi(E_{\check{n}})=\{E_n(\omega)\}_{\omega\in\Omega}$ for each $n\in\mathrm{N}$. Then, due to Proposition 4.15 and Corollary 10.4, we can see easily that $\{E_n(\omega)\}_{n\in\mathrm{N}}$ is a decreasing sequence of finite projections of $\mathscr{M}(\omega)$ with central support I with $E_n(\omega)\sim E_n(\omega)-E_{n+1}(\omega)$ for any $n\in\mathrm{N}$ almost everywhere on Ω. Therefore $\mathscr{M}(\omega)$ is continuous for almost all $\omega\in\Omega$. ■

From the preceding theorems in this section and Theorem 9.2, we have the following result.

Theorem 10.6. If $\mathscr{M}$ is a von Neumann algebra of type i acting on a separable complex Hilbert space in $V^{(\mathbf{B})}$, then $\mathscr{M}(\omega)$ is a von Neumann algebra of type i for almost all $\omega\in\Omega$ ($i=\mathrm{I}, \mathrm{II}, \mathrm{II}_\infty$).

From now on we assume that $\mathscr{M}$ is a factor in $V^{(\mathbf{B})}$, which implies, by Theorem 8.4, that $\mathscr{M}(\omega)$ is a factor for almost all $\omega\in\Omega$. In the proof of the following theorem we use two of the deepest results on the so-called Effros Borel structure attributed to Effros (1965) and Nielsen (1973).

Theorem 10.7. If $\mathscr{M}$ is of type III in $V^{(\mathbf{B})}$, then $\mathscr{M}(\omega)$ is of type III for almost all $\omega\in\Omega$.

Proof. Let $X=\{\omega\in\Omega \mid \mathscr{M}(\omega)$ is a semifinite factor$\}$. By Theorems 17.1 and 21.1 of Nielsen (1973), X is a Borel set of Ω. Then, by the semifiniteness theorem we have that $0 = [\![0\mathscr{M}$ is semifinite$]\!] \geq \Psi(X)$. Therefore X must be meager. ■

By combining Theorems 10.6 and 10.7, we have the following result.

Theorem 10.8. $\mathscr{M}$ is of type i in $V^{(\mathbf{B})}$ iff $\mathscr{M}(\omega)$ is of type i for almost all $\omega \in \Omega$ ($i = \mathrm{I}, \mathrm{II}, \mathrm{II}_\infty, \mathrm{III}$).

11. AW^*-MODULES AND AW^*-ALGEBRAS

Let $\mathbb{Z}$ be a commutative AW^*-algebra whose complete Boolean algebra is **B** and which shall be fixed throughout this section. Then $\mathbb{Z}$ can be identified with the bounded part of complex numbers in $V^{(\mathbf{B})}$. Ozawa (1984) has shown that the bounded part $\mathscr{H}_\infty^{(\mathbf{B})}$ of every complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ is an AW^*-module over $\mathbb{Z}$, which gives a bijective correspondence between the isomorphism classes of complex Hilbert spaces in $V^{(\mathbf{B})}$ and the isomorphism classes of AW^*-modules over $\mathbb{Z}$. Similarly Ozawa (1984) has shown that the bounded part $\mathscr{A}_\infty^{(\mathbf{B})}$ of every von Neumann algebra $\mathscr{A}$ acting on a complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ is a $\mathbb{Z}$-von Neumann algebra acting on the AW^*-module $\mathscr{H}_\infty^{(\mathbf{B})}$ obtained from the complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ in the above fashion, which gives an essentially bijective correspondence between the von Neumann algebras acting on the complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$ and the $\mathbb{Z}$-von Neumann algebras acting on the AW^*-module $\mathscr{H}_\infty^{(\mathbf{B})}$.

An AW^*-module X over $\mathbb{Z}$ is called *$\mathbb{Z}$-separable* if there exists a countable family $\{x_n\}_{n \in \mathrm{N}} \subset X$ such that for any $x \in X$ and any positive number ε, there exists a partition $\{b_n\}_{n \in \mathrm{N}}$ of unity of **B** with $\|b_n x - b_n x_n\| < \varepsilon$ for any $n \in N$. It is easy to see the following result.

Lemma 11.1. For any complex Hilbert space $\mathscr{H}$ in $V^{(\mathbf{B})}$, $\mathscr{H}$ is separable in $V^{(\mathbf{B})}$ iff $\mathscr{H}_\infty^{(\mathbf{B})}$ is $\mathbb{Z}$-separable.

With these correspondence results in mind, our reduction theory for separable complex Hilbert spaces and von Neumann algebras acting on them in $V^{(\mathbf{B})}$ renders immediately a reduction theory for Z-separable AW^*-modules over $\mathbb{Z}$ and $\mathbb{Z}$-von Neumann algebras acting on them. In particular, the results in Sections 8–10 still hold in this AW^*-context. Although algebraic reduction theory for AW^*-algebras has been studied by many authors (see, e.g., Berberian, 1972), it applies only to *finite* AW^*-algebras. Our reduction theory has no preference with regard to types, but applies only to embeddable AW^*-algebras satisfying a certain milder separability condition. Such a *spatial* reduction theory for AW^*-algebras as ours seems completely new.

12. CONCLUDING REMARKS

Theorem 9.1 was proved under the factor assumption. The problem of whether Theorem 9.1 still holds for general von Neumann algebras $\mathscr{M}$ acting on a separable complex Hilbert space in $V^{(\mathbf{B})}$ remains open. If the answer is affirmative, Theorem 9.2 will hold without the factor assumption and the same method will probably enable us to drop the factor assumption in Theorem 10.8 as well as Theorem 10.7. What is needed is a Boolean-valued version of Lance (1976), which might be a sort of iterated forcing.

Another interesting topic for future study is to extend our duality established in this paper to the nonseparable case. As far as complex Hilbert spaces are concerned, the problem is not difficult. Indeed, since Ozawa (1984) has already established the general duality between AW^*-modules over commutative AW^*-algebras and Boolean-valued complex Hilbert spaces, Takemoto (1973) could be regarded as a solution. However, we have no idea of how to extend this approach to von Neumann algebras.

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